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ScienceDirect
Advances in Mathematics 228 (2011) 1631–1688
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

$BMO_L(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators

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Received 1 March 2009; accepted 1 June 2011

Available online 16 July 2011

Communicated by Erwin Lutwak

Abstract

Let $L = -\Delta_{\mathbb{H}^n} + V$ be a Schrödinger operator on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and the nonnegative potential V belongs to the reverse Hölder class $B_{Q/2}$. Here Q is the homogeneous dimension of $\mathbb{H}^n$. In this article we investigate the dual space of the Hardy-type space $H_L^1(\mathbb{H}^n)$ associated with the Schrödinger operator L, which is a kind of BMO-type space $BMO_L(\mathbb{H}^n)$ defined by means of a revised sharp function related to the potential V. We give the Fefferman–Stein type decomposition of BMO_L-functions with respect to the (adjoint) Riesz transforms $\widetilde{R}_j^L$ for L, and characterize $BMO_L(\mathbb{H}^n)$ in terms of the Carleson measure. We also establish the BMO_L-boundedness of some operators, such as the (adjoint) Riesz transforms $\widetilde{R}_j^L$, the Littlewood–Paley function s_Q^L, the Lusin area integral S_Q^L, the Hardy–Littlewood maximal function, and the semigroup maximal function. All results hold for stratified groups as well.

MSC: 22E30; 35J10; 42B30; 43A80

Keywords: *BMO* space; Carleson measure; Hardy space; Heisenberg group; Reverse Hölder class; Schrödinger operator; Stratified group

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1 Supported by National Science Council of Taiwan under Grant #NSC 97-2115-M-008-021-MY3.
2 Supported by National Natural Science Foundation of China under Grant #10871003, #10990012 and the Specialized Research Fund for the Doctoral Program of Higher Education of China under Grant #2007001040.

doi:10.1016/j.aim.2011.06.024

1. Introduction

Let $L = -\Delta_{\mathbb{H}^n} + V$ be a Schrödinger operator on the Heisenberg group $\mathbb{H}^n$. Here $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and the nonnegative potential V belongs to the reverse Hölder class $B_{Q/2}$ and Q is the homogeneous dimension of $\mathbb{H}^n$. The Hardy-type space $H_L^1(\mathbb{H}^n)$ associated with the Schrödinger operator L has been studied in [14], which is defined by means of the maximal function with respect to the semigroup $\{e^{-sL}\colon s > 0\}$. The space $H_L^1(\mathbb{H}^n)$ can be characterized by the atomic decomposition or by the Riesz transforms for the Schrödinger operator L given by $R_j^L = X_j L^{-\frac{1}{2}}$, $j = 1, \ldots, 2n$, where the X_j's are left-invariant vector fields that generate the Lie algebra of $\mathbb{H}^n$. In this article we investigate the dual space of $H_L^1(\mathbb{H}^n)$, which is a *BMO*-type space defined by means of a revised sharp function related to the potential V. The Riesz transforms R_j^L may be unbounded on $L^p(\mathbb{H}^n)$ if $p > Q$ (cf. [14]). We prove the adjoint Riesz transforms $\widetilde{R}_j^L = L^{-\frac{1}{2}} X_j$ are bounded on $BMO_L(\mathbb{H}^n)$ and give the Fefferman–Stein type decomposition of BMO_L-functions with respect to $\widetilde{R}_j^L$. We characterize $BMO_L(\mathbb{H}^n)$ in terms of the Carleson measure. In order to get such a characterization we investigate the Littlewood–Paley function s_Q^L and the Lusin area integral S_Q^L related to L. These operators are bounded on $BMO_L(\mathbb{H}^n)$, bounded from $H_L^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$, and bounded from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$. We also prove that the Hardy–Littlewood maximal function and the maximal function with respect to the semigroup $\{e^{-sL}\colon s > 0\}$ are bounded on $BMO_L(\mathbb{H}^n)$.

The *BMO*-type space BMO_L associated with a Schrödinger operator L on the Euclidean space was investigated by Dziubański et al. [7]. The corresponding results for $BMO(G)$ and $H^1(G)$ are established in [9], and the Fefferman–Stein type decomposition of $BMO(G)$-functions is proved by Christ and Geller [4], where G is a homogeneous groups. This article extends the results of [7] and other known results on the Euclidean space to the Heisenberg group. One of the building block for the results such as BMO_L-boundedness of maximal functions in the Euclidean case was the Calderón–Zygmund decomposition. Despite the fact that there is a version of the Calderón–Zygmund decomposition for homogeneous spaces [3] (see also [16]) and it is very useful for some problems, due to the intricacy between the Euclidean setting and ours, this version does not capture all the essential ingredients that is vital for us and thus offers no help to us at this moment. We propose a new method of obtaining the estimates of BMO_L-boundedness of maximal functions in the Heisenberg group that bypasses altogether the use of any version of the Calderón–Zygmund decomposition. As a consequence of our new approach, we also obtain the BMO_L-boundedness of the adjoint Riesz transforms $\widetilde{R}_j^L$ and the Lusin area integral S_Q^L related to L. These two results are new even in the case of Euclidean spaces. Furthermore, all our results hold for stratified groups.

This article is organized as follows. Basic definitions, notations, and our main results are stated in Section 2. In Section 3, we give some estimates of the kernels related to the Schrödinger operator L. In Section 4, we establish the duality of $H_L^1(\mathbb{H}^n)$ and $BMO_L(\mathbb{H}^n)$. In Section 5, we prove the BMO_L-boundedness of $\widetilde{R}_j^L$ and get the Fefferman–Stein type decomposition of BMO_L-functions with respect to $\widetilde{R}_j^L$. Section 6 is devoted to investigating the Littlewood–Paley function s_Q^L and the Lusin area integral S_Q^L. The characterization of $BMO_L(\mathbb{H}^n)$ in terms of the Carleson measure is given in Section 7. The BMO_L-boundedness of maximal functions and square functions is proved in Section 8. Finally, we include in Section 9 a brief discussion the corresponding results for stratified groups without proofs. Throughout the article, we will use A and C to denote positive constants, which are independent of the main parameters and may be different at each occurrence.

2. Notations and main results

The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is a nilpotent Lie group with underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$. The group structure is given by

$$(x,t)(y,s) = \left(x+y, t+s+2\sum_{j=1}^{n}(x_{n+j}y_j - x_j y_{n+j})\right).$$

The Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is spanned by

$$X_{2n+1} = \frac{\partial}{\partial t}, \qquad X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t}, \qquad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, \dots, n.$$

All non-trivial commutation relations are given by $[X_j, X_{n+j}] = -4X_{2n+1}$, $j = 1, \dots, n$. The sub-Laplacian $\Delta_{\mathbb{H}^n}$ and the gradient $\nabla_{\mathbb{H}^n}$ are defined respectively by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2 \quad \text{and} \quad \nabla_{\mathbb{H}^n} = (X_1, \dots, X_{2n}).$$

The dilations on $\mathbb{H}^n$ have the form

$$\delta_r(x,t) = (rx, r^2 t), \quad r > 0.$$

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. The measure of any measurable set E is denoted by $|E|$. We define a homogeneous norm on $\mathbb{H}^n$ by

$$|g| = \left(|x|^4 + |t|^2\right)^{\frac{1}{4}}, \quad g = (x,t) \in \mathbb{H}^n.$$

This norm satisfies the triangle inequality and leads to a left-invariant distant $d(g,h) = |g^{-1}h|$. The ball of radius r centered at g is denoted by

$$B(g,r) = \left\{h \in \mathbb{H}^n\colon \left|g^{-1}h\right| < r\right\},$$

whose volume is given by

$$\left|B(g,r)\right| = c_n r^Q, \quad c_n = \left|B(0,1)\right| = \frac{2\pi^{n+\frac{1}{2}}\Gamma(\frac{n}{2})}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})},$$

and $Q = 2n+2$ the homogeneous dimension of $\mathbb{H}^n$.

A nonnegative locally L^q integrable function V on $\mathbb{H}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(h)^q\,dh\right)^{\frac{1}{q}} \leqslant C\left(\frac{1}{|B|}\int_B V(h)\,dh\right)$$

holds for every ball B in $\mathbb{H}^n$. In this article we always assume that $0 \not\equiv V \in B_{Q/2}$.

We consider the Schrödinger operator $L = -\Delta_{\mathbb{H}^n} + V$. Since $V \geqslant 0$ and $V \in L^{Q/2}_{\mathrm{loc}}(\mathbb{H}^n)$, L generates a (C_0) contraction semigroup $\{T_s^L\colon s>0\} = \{e^{-sL}\colon s>0\}$. We defined the semigroup maximal function related to L by

$$T_L^* f(g) = \sup_{s>0}\left|T_s^L f(g)\right|, \quad g \in \mathbb{H}^n.$$

The Hardy space $H^1_L(\mathbb{H}^n)$ associated with the Schrödinger operator L is defined to be

$$H^1_L\left(\mathbb{H}^n\right) = \left\{f \in L^1\left(\mathbb{H}^n\right)\colon\ T_L^* f \in L^1\left(\mathbb{H}^n\right)\right\}$$

with

$$\|f\|_{H^1_L} = \left\|T_L^* f\right\|_{L^1}.$$

In order to define an atom in $H^1_L(\mathbb{H}^n)$, we introduce the auxiliary function $\rho(g,V) = \rho(g)$ defined by

$$\rho(g) = \sup_{r>0}\left\{r\colon\ \frac{1}{r^{Q-2}}\int_{B(g,r)} V(h)\,dh \leqslant 1\right\}, \quad g \in \mathbb{H}^n.$$

It is known that $0 < \rho(g) < \infty$ for any $g \in \mathbb{H}^n$ (by Lemma 2 in Section 3). Let $1 < q \leqslant \infty$. A function a is called an $H^{1,q}_L$*-atom* (*centered at* g) if it satisfies the following conditions:

(i) $\operatorname{supp}(a) \subset B(g,r)$,
(ii) $\|a\|_{L^q} \leqslant |B(g,r)|^{\frac{1}{q}-1}$,
(iii) if $r < \rho(g)$, then $\int_{B(g,r)} a(h)\,dh = 0$.

Then $H^1_L(\mathbb{H}^n)$ admits an atomic characterization as follows (cf. [14]).

Proposition 1. *Let $f \in L^1(\mathbb{H}^n)$ and $1 < q \leqslant \infty$. Then $f \in H^1_L(\mathbb{H}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where the a_j's are $H^{1,q}_L$-atoms, $\sum_j |\lambda_j| < \infty$, and the sum converges in $H^1_L(\mathbb{H}^n)$ norm. Moreover,*

$$\|f\|_{H^1_L} \sim \inf\left\{\sum_j |\lambda_j|\right\},$$

where the infimum is taken over all atomic decompositions of f into $H^{1,q}_L$-atoms. As usual, here by $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{B_2}{C} \leqslant B_1 \leqslant C B_2$.

Let φ be a locally integrable function on $\mathbb{H}^n$. $B = B(g,r)$ is a ball centered at g. Set

$$\varphi(B) = \frac{1}{|B(g,r)|} \int\limits_{B(g,r)} \varphi(h)\, dh$$

and

$$\varphi(B,V) = \begin{cases} \varphi(B), & \text{if } r < \rho(g), \\ 0, & \text{if } r \geqslant \rho(g). \end{cases}$$

We define the revised sharp function related to the potential V by

$$\varphi_V^\sharp(g) = \sup_{g\in B} \frac{1}{|B|} \int\limits_B \big|\varphi(h) - \varphi(B,V)\big|\, dh.$$

Now we define the space $BMO_L(\mathbb{H}^n)$ associated with the Schrödinger operator L.

Definition 1. Let φ be a locally integrable function on $\mathbb{H}^n$. If $\varphi_V^\sharp \in L^\infty(\mathbb{H}^n)$, then we say $\varphi \in BMO_L(\mathbb{H}^n)$ and set $\|\varphi\|_{BMO_L} = \|\varphi_V^\sharp\|_{L^\infty}$.

Remark 1. A function $\varphi \in BMO_L(\mathbb{H}^n)$ if and only if there exists a scalar C_B (depending on $B = B(g,r)$ and satisfying $C_B = 0$ whenever $r \geqslant \rho(g)$) such that

$$\sup_B \frac{1}{|B|} \int\limits_B \big|\varphi(h) - C_B\big|\, dh < \infty.$$

It is clear that $L^\infty(\mathbb{H}^n) \subset BMO_L(\mathbb{H}^n) \subset BMO(\mathbb{H}^n)$ and $\|\varphi\|_{BMO} \leqslant 2\|\varphi\|_{BMO_L}$. We note that if $\varphi \in BMO(\mathbb{H}^n)$, then

$$\int\limits_{\mathbb{H}^n} \frac{|\varphi(g)|}{(1+|g|)^{Q+1}}\, dg < \infty \tag{1}$$

(cf. [9, Proposition (5.9)]). Also $\|\varphi\|_{BMO_L} = 0$ if and only if $\varphi(g) = 0$ for almost every $g \in \mathbb{H}^n$.

Let L_c^∞ denote the space of all bounded functions with compact supports. It is clear that L_c^∞ is exactly the space of finite linear combinations of $H_L^{1,\infty}$-atoms. By Proposition 1, L_c^∞ is a dense subspace of $H_L^1(\mathbb{H}^n)$. Set

$$\mathcal{L}_\varphi(f) = \int\limits_{\mathbb{H}^n} f(g)\varphi(g)\, dg, \quad f \in L_c^\infty,\ \varphi \in L^1_{\mathrm{loc}}\big(\mathbb{H}^n\big). \tag{2}$$

We will prove the following theorem.

Theorem 1.

(a) *Suppose* $\varphi \in BMO_L(\mathbb{H}^n)$. *Then* $\mathcal{L}_\varphi$ *given by* (2) *extends to a bounded linear functional on* $H_L^1(\mathbb{H}^n)$ *and satisfies*

$$\|\mathcal{L}_\varphi\| \leqslant C\|\varphi\|_{BMO_L}.$$

(b) *Conversely, every bounded linear functional* $\mathcal{L}$ *on* $H_L^1(\mathbb{H}^n)$ *can be realized as* $\mathcal{L} = \mathcal{L}_\varphi$ *with* $\varphi \in BMO_L(\mathbb{H}^n)$ *and*

$$\|\varphi\|_{BMO_L} \leqslant C\|\mathcal{L}\|.$$

Let us consider the *Riesz transforms* R_j^L and the *adjoint Riesz transforms* $\widetilde{R}_j^L$ for the Schrödinger operator L defined by

$$\begin{cases} R_j^L = X_j L^{-\frac{1}{2}}, \\ \widetilde{R}_j^L = L^{-\frac{1}{2}} X_j = -\left(R_j^L\right)^*, \end{cases} \qquad j = 1, \ldots, 2n.$$

R_j^L are bounded on $L^p(\mathbb{H}^n)$ for $1 < p \leqslant Q$ (cf. [13]). This is equivalent to the boundedness of $\widetilde{R}_j^L$ on $L^{p'}(\mathbb{H}^n)$, $\frac{Q}{Q-1} \leqslant p' < \infty$. The range of p in the above is optimal (cf. [14]). $H_L^1(\mathbb{H}^n)$ is also characterized by R_j^L as follows (cf. [14]).

Proposition 2. *A function* $f \in H_L^1(\mathbb{H}^n)$ *if and only if* $f \in L^1(\mathbb{H}^n)$ *and* $R_j^L f \in L^1(\mathbb{H}^n)$, $j = 1, \ldots, 2n$. *Moreover,*

$$\|f\|_{H_L^1} \sim \|f\|_{L^1} + \sum_{j=1}^{2n} \left\|R_j^L f\right\|_{L^1}.$$

Our next result is stated as follows.

Theorem 2.

(a) $\widetilde{R}_j^L$ *are bounded on* $BMO_L(\mathbb{H}^n)$.
(b) $\varphi \in BMO_L(\mathbb{H}^n)$ *if and only if there exist* $\varphi_0, \varphi_1, \ldots, \varphi_{2n} \in L^\infty(\mathbb{H}^n)$ *such that*

$$\varphi(g) = \varphi_0(g) + \sum_{j=1}^{2n} \widetilde{R}_j^L \varphi_j(g).$$

The proof of Theorem 2 shows that $\widetilde{R}_j^L$ are definitely defined on $BMO_L(\mathbb{H}^n)$ without the ambiguity of an additive constant. The analogue of Theorem 2 on the Euclidean space is also true and can be proved by the same argument.

Remark 2. Theorem 2(a) does not hold for Riesz transforms R_j^L. If R_j^L were bounded on $BMO_L(\mathbb{H}^n)$, then R_j^L would be bounded from $L^\infty(\mathbb{H}^n)$ into $BMO(\mathbb{H}^n)$ since $L^\infty(\mathbb{H}^n) \subset BMO_L(\mathbb{H}^n) \subset BMO(\mathbb{H}^n)$. On the other hand, Proposition 2 implies the $H^1(\mathbb{H}^n) - L^1(\mathbb{H}^n)$ boundedness of R_j^L. By interpolation, we would obtain the $L^p(\mathbb{H}^n)$ boundedness of R_j^L for $1 < p < \infty$, which is not true.

We will characterize $BMO_L(\mathbb{H}^n)$ in terms of the Carleson measure. Let $\mathbf{U}^n$ be the Siegel upper half-space in $\mathbb{C}^{n+1}$, i.e.,

$$\mathbf{U}^n = \left\{ z \in \mathbb{C}^{n+1}\colon \operatorname{Im} z_{n+1} > \sum_{j=1}^n |z_j|^2 \right\}.$$

Then $\mathbf{U}^n$ is holomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$. It is well known that the Heisenberg group $\mathbb{H}^n$ is a nilpotent subgroup of the automorphism group of $\mathbf{U}^n$, which consists of the translations of $\mathbf{U}^n$. The Heisenberg group $\mathbb{H}^n$ can also be identified with the boundary $\partial\mathbf{U}^n$ via its action on the origin (cf. [18, p. 531]). We use the Heisenberg coordinates $(g, s) = (x, t, s)$ to denote the points in $\mathbf{U}^n$, where

$$x_j + i x_{n+j} = z_j, \quad j = 1, \dots, n, \qquad t = \operatorname{Re} z_{n+1}, \qquad s = \operatorname{Im} z_{n+1} - \sum_{j=1}^n |z_j|^2.$$

For any ball $B = B(g, r)$ in $\mathbb{H}^n$, we define the *Carleson box* $\Omega(B) = \Omega(g, r)$ based on B by

$$\Omega(g, r) = \left\{ (h, s) \in \mathbf{U}^n\colon \left|g^{-1}h\right| < r,\ 0 < s < r^2 \right\}.$$

A nonnegative Borel measure μ on $\mathbf{U}^n$ is called a *Carleson measure* if

$$\|\mu\|_{\mathcal{C}} = \sup_B \frac{\mu(\Omega(B))}{|B|} < \infty.$$

Let

$$Q_s^L \varphi(g) = s \frac{d}{ds} T_s^L \varphi(g), \quad g \in \mathbb{H}^n,\ s > 0.$$

Then $Q_s^L \varphi$ is well defined if φ satisfies (1) (see Lemma 12 in Section 2). We obtain a nonnegative Borel measure $d\mu_\varphi$ on $\mathbf{U}^n$ defined by

$$d\mu_\varphi(g, s) = \left| Q_s^L \varphi(g) \right|^2 \frac{dg\, ds}{s}, \quad (g, s) \in \mathbf{U}^n.$$

Theorem 3.

(a) *If $\varphi \in BMO_L(\mathbb{H}^n)$, then $d\mu_\varphi$ is a Carleson measure with*

$$\|d\mu_\varphi\|_{\mathcal{C}} \leqslant C \|\varphi\|_{BMO_L}^2.$$

(b) *Conversely, if φ satisfies* (1) *and $d\mu_\varphi$ is a Carleson measure, then $\varphi \in BMO_L(\mathbb{H}^n)$ and*

$$\|\varphi\|_{BMO_L}^2 \leqslant C\|d\mu_\varphi\|_{\mathcal{C}}.$$

In order to establish Theorem 3, we investigate the Littlewood–Paley function s_Q^L and the Lusin area integral S_Q^L related to L, which are defined respectively by

$$s_Q^L f(g) = \left(\int_0^\infty \left|Q_s^L f(g)\right|^2 \frac{ds}{s} \right)^{\frac{1}{2}}$$

and

$$S_Q^L f(g) = \left(\int_{\Gamma(g)} \left|Q_s^L f(h)\right|^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}} \right)^{\frac{1}{2}},$$

where

$$\Gamma(g) = \left\{(h,s) \in \mathbf{U}^n\colon \left|g^{-1}h\right| < \sqrt{s}\right\}.$$

Theorem 4. *The operators s_Q^L and S_Q^L are bounded from $H_L^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$ and bounded from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$. For $1 < p < \infty$,*

$$\left\|s_Q^L f\right\|_{L^p} \sim \left\|S_Q^L f\right\|_{L^p} \sim \|f\|_{L^p}.$$

The theory of tent spaces on the half-space $\mathbb{R}_+^{n+1}$ is established by Coifman, Meyer and Stein [5]. We extend the duality inequality of tent spaces to the Siegel upper half-space $\mathbf{U}^n$, which is useful in the proof of Theorem 3. Let $F(g,s)$ and $\Phi(g,s)$ be measurable functions on $\mathbf{U}^n$. We set

$$\mathcal{A}(F)(g) := \left(\int_{\Gamma(g)} \left|F(h,s)\right|^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}} \right)^{\frac{1}{2}},$$

$$\mathcal{C}(\Phi)(g) := \sup_{g\in B}\left(\frac{1}{|B|} \int_{\Omega(B)} \left|\Phi(h,s)\right|^2 \frac{dh\,ds}{s} \right)^{\frac{1}{2}}.$$

Theorem 5. *Let $F(g,s)$ and $\Phi(g,s)$ be measurable functions on $\mathbf{U}^n$ such that $\mathcal{A}(F) \in L^1(\mathbb{H}^n)$ and $\mathcal{C}(\Phi) \in L^\infty(\mathbb{H}^n)$. Then we have the following duality inequality.*

$$\begin{aligned}\int_{\mathbf{U}^n} \left|F(g,s)\Phi(g,s)\right| \frac{dg\,ds}{s} &\leqslant C \int_{\mathbb{H}^n} \mathcal{A}(F)(g)\mathcal{C}(\Phi)(g)\,dg \\ &\leqslant C\left\|\mathcal{A}(F)\right\|_{L^1}\left\|\mathcal{C}(\Phi)\right\|_{L^\infty}.\end{aligned}$$

We will also prove the BMO_L-boundedness of maximal functions and square functions. Let Mf denote the Hardy–Littlewood maximal function defined by

$$Mf(g) = \sup_{g \in B} \frac{1}{|B|} \int_B |f(h)| \, dh.$$

Theorem 6. *The Hardy–Littlewood maximal function M and the semigroup maximal function T_L^* are bounded on $BMO_L(\mathbb{H}^n)$. The Littlewood–Paley function s_Q^L and the Lusin area integral S_Q^L are bounded on $BMO_L(\mathbb{H}^n)$.*

Remark 3. A same argument as [18, p. 57, (16)] yields $T_L^* f \leqslant CMf$. The operators mentioned in Theorem 6 are all bounded from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$, and hence their $L^p(\mathbb{H}^n)$, $1 < p < \infty$, boundedness can be directly obtained by interpolation between $L^{1,\infty}(\mathbb{H}^n)$ and $BMO_L(\mathbb{H}^n)$.

The BMO_L-boundedness of the Hardy–Littlewood maximal function, the semigroup maximal function, and the Littlewood–Paley function s_Q^L on the Euclidean space was established by Dziubański et al. [7]. The approach in [7] relied on a result of Bennet, DeVore and Sharpley [1] which states that for a function $f \in BMO$, the corresponding Hardy–Littlewood maximal function Mf is either identically equals to infinity or Mf belongs to BMO. Bennet, DeVore and Sharpley used the precise dyadic decomposition of the Calderón–Zygmund decomposition to establish such a result. One would attempt to repeat the argument here but soon discovers that the situation in the Heisenberg group is much more delicate. That is to say, even Christ's version of the Calderón–Zygmund decomposition [3] is not applicable here. Therefore, in order to establish the corresponding result on the Heisenberg group, we establish the relevant estimates directly. As a consequence of our approach, we also obtain a new result not previously known in the Euclidean case: the BMO_L-boundedness of the Lusin area integral S_Q^L.

3. Estimates of the kernels

We first collect some basic facts about the potential V satisfying the reverse Hölder inequality. Obviously, $B_{q_1} \subset B_{q_2}$ if $q_1 > q_2$. It is important that the B_q class has a property of "self-improvement"; that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$. We have assumed that $V \in B_{\frac{Q}{2}}$, and hence $V \in B_{q_0}$ for some q_0 satisfying $\frac{Q}{2} < q_0 < Q$. We also write $\delta = 2 - \frac{Q}{q_0} \in (0, 1)$, and throughout the paper we keep this assumption and the meanings of q_0 and δ.

Lemma 1. *The measure $V(h)\,dh$ satisfies the doubling condition; that is, there exists $C_0 > 0$ such that*

$$\int_{B(g,2r)} V(h)\,dh \leqslant C_0 \int_{B(g,r)} V(h)\,dh$$

for all balls $B(g, r)$ in $\mathbb{H}^n$.

Lemma 2. *For* $0 < r < R < \infty$,

$$\frac{1}{r^{Q-2}} \int\limits_{B(g,r)} V(h)\,dh \leqslant C\left(\frac{r}{R}\right)^{\delta} \frac{1}{R^{Q-2}} \int\limits_{B(g,R)} V(h)\,dh.$$

Lemma 3. *If* $r = \rho(g)$, *then*

$$\frac{1}{r^{Q-2}} \int\limits_{B(g,r)} V(h)\,dh = 1.$$

Moreover,

$$\frac{1}{r^{Q-2}} \int\limits_{B(g,r)} V(h)\,dh \sim 1 \quad \textit{if and only if} \quad r \sim \rho(g).$$

Lemma 4. *There exists* $m_0 > 0$ *such that, for any* g *and* h *in* $\mathbb{H}^n$,

$$\frac{1}{C}\left(1 + \frac{|h^{-1}g|}{\rho(g)}\right)^{-m_0} \leqslant \frac{\rho(h)}{\rho(g)} \leqslant C\left(1 + \frac{|h^{-1}g|}{\rho(g)}\right)^{\frac{m_0}{m_0+1}}.$$

In particular, $\rho(h) \sim \rho(g)$ *if* $|h^{-1}g| < C\rho(g)$.

Lemma 5. *There exists* $l_0 > 1$ *such that*

$$\int\limits_{B(g,R)} \frac{V(h)}{|h^{-1}g|^{Q-2}}\,dh \leqslant \frac{C}{R^{Q-2}} \int\limits_{B(g,R)} V(h)\,dh \leqslant C\left(1 + \frac{R}{\rho(g)}\right)^{l_0}.$$

For Lemmas 1–5, we refer readers to [15].

We say that a function φ has *rapid decay* if, for any $N > 0$, there exists a constant $C_N > 0$ such that

$$\left|\varphi(g)\right| \leqslant C_N\left(1 + |g|\right)^{-N}.$$

Let $\varphi_s(g) = s^{-\frac{Q}{2}}\varphi(\delta_{\frac{1}{\sqrt{s}}}g)$.

Lemma 6. *Suppose* φ *is a nonnegative function with rapid decay. Then*

$$\int\limits_{\mathbb{H}^n} V(h)\varphi_s\left(h^{-1}g\right)dh \leqslant \begin{cases} \frac{C}{s}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta} & \textit{if } s < \rho(g)^2, \\ \frac{C}{s}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{l_0} & \textit{if } s \geqslant \rho(g)^2, \end{cases}$$

where l_0 *is given in Lemma* 5.

Proof. Write

$$\begin{aligned}\int\limits_{\mathbb{H}^n} V(h)\varphi_s\big(h^{-1}g\big)\,dh &= \int\limits_{|h^{-1}g|<\sqrt{s}} V(h)\varphi_s\big(h^{-1}g\big)\,dh + \int\limits_{|h^{-1}g|\geqslant\sqrt{s}} V(h)\varphi_s\big(h^{-1}g\big)\,dh \\ &\leqslant \frac{C}{s}\frac{1}{(\sqrt{s})^{Q-2}} \int\limits_{|h^{-1}g|<\sqrt{s}} V(h)\,dh \\ &\quad + \frac{C_N}{s}\sum_{k=0}^{\infty} 2^{-kN}\frac{1}{(\sqrt{s})^{Q-2}} \int\limits_{2^k\sqrt{s}\leqslant|h^{-1}g|<2^{k+1}\sqrt{s}} V(h)\,dh \\ &= I_1 + I_2.\end{aligned}$$

If $s < \rho(g)^2$, making use of Lemmas 1 to 3, we get

$$I_1 \leqslant \frac{C}{s}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta}$$

and

$$I_2 \leqslant \frac{C_N}{s}\sum_{k=0}^{\infty}\left(\frac{C_0}{2^N}\right)^k \frac{1}{(\sqrt{s})^{Q-2}} \int\limits_{|h^{-1}g|<\sqrt{s}} V(h)\,dh \leqslant \frac{C}{s}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta}$$

provided N large enough.

If $s \geqslant \rho(g)^2$, by Lemma 5 we have

$$I_1 \leqslant \frac{C}{s}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{l_0}$$

and

$$I_2 \leqslant \frac{C_N}{s}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{l_0}\sum_{k=0}^{\infty} 2^{-k(N-Q+2-l_0)} \leqslant \frac{C}{s}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{l_0}$$

provided N large enough. □

Now we turn to the estimates of the kernels related to the Schrödinger operator L. We denote by $H_s(g)$ the convolution kernel of heat semigroup $\{T_s\colon s>0\} = \{e^{s\Delta_{\mathbb{H}^n}}\colon s>0\}$. The heat kernel $H_s(g)$ satisfies the estimate (cf. [11])

$$0 < H_s(g) \leqslant C s^{-\frac{Q}{2}} e^{-As^{-1}|g|^2}. \tag{3}$$

Let $K_s^L(g,h)$ denote the kernel of T_s^L. Since $V \geqslant 0$, by the Trotter product formula (cf. [10, p. 53]) and (3),

$$0 \leqslant K_s^L(g,h) \leqslant H_s\big(h^{-1}g\big) \leqslant C s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}. \tag{4}$$

$\{T_s^L\}$ extends to a holomorphic semigroup $\{T_\zeta^L\}$ on $L^2(\mathbb{H}^n)$ for $\operatorname{Re}\zeta > 0$ and

$$\big\|T_\zeta^L\big\|_{L^2\mapsto L^2} \leqslant 1 \quad \text{for } \operatorname{Re}\zeta \geqslant 0 \tag{5}$$

(cf. [17, Chapter 3, Theorem 1]). Let $K_\zeta^L(g,h)$ denote the kernel of T_ζ^L. The next two lemmas give the estimates of $K_s^L(g,h)$ and $K_\zeta^L(g,h)$, which are more accurate than (4).

Lemma 7. *For any $N > 0$, there exists a constant $C_N > 0$ such that*

$$0 \leqslant K_s^L(g,h) \leqslant C_N s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(1 + \frac{\sqrt{s}}{\rho(g)} + \frac{\sqrt{s}}{\rho(h)}\right)^{-N}.$$

Proof. Since $K_{s+i\tau}^L(g,h) = T_{i\tau}^L(K_s^L(\cdot,h))(g)$ for $s > 0$ and $\tau \in \mathbb{R}$, it follows from (4) and (5) that

$$\int_{\mathbb{H}^n} \big|K_{s+i\tau}^L(g,h)\big|^2\,dg \leqslant \int_{\mathbb{H}^n} K_s^L(g,h)^2\,dg \leqslant C s^{-\frac{Q}{2}},$$

which yields

$$\begin{aligned}\big|K_{s+i\tau}^L(g,h)\big| &= \left|\int_{\mathbb{H}^n} K_{\frac{s}{2}}^L(g,w) K_{\frac{s}{2}+i\tau}^L(w,h)\,dw\right| \\ &\leqslant \left(\int_{\mathbb{H}^n} K_{\frac{s}{2}}^L(g,w)^2\,dw\right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^n} \big|K_{\frac{s}{2}+i\tau}^L(w,h)\big|^2\,dw\right)^{\frac{1}{2}} \leqslant C s^{-\frac{Q}{2}}.\end{aligned}$$

The Cauchy integral formula gives

$$\big|\partial_s^N K_s^L(g,h)\big| = \left|\frac{N!}{2\pi i}\int_{|\zeta-s|=\frac{s}{2}} \frac{K_\zeta^L(g,h)}{(\zeta-s)^{N+1}}\,d\zeta\right| \leqslant C_N s^{-N-\frac{Q}{2}}. \tag{6}$$

Let $\Gamma^L(g,h)$ denote the fundamental solution for L. It is known that for any $j > 0$, there exists a constant $C_j > 0$ such that

$$0 \leqslant \Gamma^L(g,h) \leqslant \frac{C_j}{(1 + |h^{-1}g|(\rho(g)^{-1} + \rho(h)^{-1}))^j |h^{-1}g|^{Q-2}} \tag{7}$$

(cf. [14, (5)] or [15, Theorem 4.8]). For any $m \in \mathbb{N}\cup\{0\}$ and $f \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, the above inequality (7) and Lemma 4 imply

$$\begin{aligned}
&\left| \int\limits_{\mathbb{H}^n} \Gamma^L(g,h)\rho(h)^m f(h)\,dh \right| \\
&\quad \leqslant C\rho(g)^m \sum_{k=-\infty}^{0} \int\limits_{2^{k-1}\rho(g)\leqslant|h^{-1}g|<2^k\rho(g)} \frac{|f(h)|}{|h^{-1}g|^{Q-2}}\,dh \\
&\qquad + C_j\rho(g)^m \sum_{k=1}^{\infty} \int\limits_{2^{k-1}\rho(g)\leqslant|h^{-1}g|<2^k\rho(g)} \frac{|f(h)|}{(1+|h^{-1}g|\rho(g)^{-1})^{j-\frac{m_0 m}{m_0+1}}|h^{-1}g|^{Q-2}}\,dh \\
&\quad \leqslant C\rho(g)^{m+2}Mf(g)\left(\sum_{k=-\infty}^{0} 2^{2k} + \sum_{k=1}^{\infty} 2^{-k(j-\frac{m_0 m}{m_0+1}-2)}\right) \leqslant C\rho(g)^{m+2}Mf(g),
\end{aligned}$$

where j is chosen large enough. By induction, we obtain

$$\left|L^{-N} f(g)\right| \leqslant C_N \rho(g)^{2N} M^N f(g) \quad \forall N \in \mathbb{N}. \tag{8}$$

It follows from (6) and (8) that

$$K_s^L(g,h) = \left|L^{-N}\partial_s^N K_s^L(g,h)\right| \leqslant C_N s^{-N-\frac{Q}{2}}\rho(g)^{2N}.$$

Because $K_s^L(g,h)$ is symmetric with respect to g and h, we have

$$K_s^L(g,h) \leqslant C_N s^{-\frac{Q}{2}}\left(\frac{\sqrt{s}}{\rho(g)} + \frac{\sqrt{s}}{\rho(h)}\right)^{-2N}. \tag{9}$$

Lemma 7 follows from (4) and (9). □

Lemma 8. *For any $N > 0$, there exists a constant $C_N > 0$ such that*

$$\left|K_\zeta^L(g,h)\right| \leqslant C_N(\operatorname{Re}\zeta)^{-\frac{Q}{2}} e^{-A(\operatorname{Re}\zeta)^{-1}|h^{-1}g|^2}\left(1 + \frac{\sqrt{\operatorname{Re}\zeta}}{\rho(g)} + \frac{\sqrt{\operatorname{Re}\zeta}}{\rho(h)}\right)^{-N}, \quad |\arg\zeta| < \frac{\pi}{4}.$$

Proof. Set $dm_{\beta,h}(g) = e^{\beta|h^{-1}g|}\,dg$, where $\beta > 0$ and $h \in \mathbb{H}^n$. By (4), we have

$$\begin{aligned}
&\left(\int\limits_{\mathbb{H}^n} \left|T_s^L f(g)\right|^2 dm_{\beta,h}(g)\right)^{\frac{1}{2}} \\
&\quad \leqslant C\left(\int\limits_{\mathbb{H}^n}\left(\int\limits_{\mathbb{H}^n} \left|f(w)\right| s^{-\frac{Q}{2}} e^{-As^{-1}|w^{-1}g|^2}\,dw\right)^2 e^{\beta|h^{-1}g|}\,dg\right)^{\frac{1}{2}} \\
&\quad \leqslant C\int\limits_{\mathbb{H}^n} s^{-\frac{Q}{2}} e^{-As^{-1}|w|^2}\left(\int\limits_{\mathbb{H}^n}\left|f(gw^{-1})\right|^2 e^{\beta|h^{-1}g|}\,dg\right)^{\frac{1}{2}} dw
\end{aligned}$$

$$\leqslant C\left(\int_{\mathbb{H}^n} s^{-\frac{Q}{2}} e^{-As^{-1}|w|^2+\frac{\beta}{2}|w|}\,dw\right)\left(\int_{\mathbb{H}^n}\left|f(g)\right|^2 e^{\beta|h^{-1}g|}\,dg\right)^{\frac{1}{2}}$$

$$\leqslant Ce^{b_1 s\beta^2}\left(\int_{\mathbb{H}^n}\left|f(g)\right|^2\,dm_{\beta,h}(g)\right)^{\frac{1}{2}},$$

where b_1 is a positive constant independent of β and h. In the rest of the proof all b_j are positive constants independent of β and h. We have proved that

$$\left\|T_s^L\right\|_{L^2(\mathbb{H}^n,dm_{\beta,h})\mapsto L^2(\mathbb{H}^n,dm_{\beta,h})} \leqslant Ce^{b_1 s\beta^2}. \tag{10}$$

Take $\varphi, \psi \in C_c^\infty(\mathbb{H}^n)$ satisfying $\|\varphi\|_{L^2} = \|\psi\|_{L^2} = 1$. We define a function $F_{\varphi,\psi}$ by

$$F_{\varphi,\psi}(z) = e^{-4b_1\beta^2 e^{i\frac{\pi}{2}z}} \int_{\mathbb{H}^n} T^L_{e^{i\frac{\pi}{2}z}}\left(e^{-(1-z)\beta|h^{-1}\cdot|}\varphi\right)(w)\left(e^{-(1-z)\beta|h^{-1}w|}\psi(w)\right)e^{2(1-z)\beta|h^{-1}w|}\,dw,$$

which is holomorphic in the strip $\{z = x+iy \in \mathbb{C}\colon\ 0 < x < 1\}$ and continuous in the closure. Since φ and ψ are compact supported, by Schwarz's inequality and (5), $|F_{\varphi,\psi}(z)| \leqslant C_\psi$. Note that

$$\left\|e^{-(1-z)\beta|h^{-1}\cdot|}\varphi\right\|_{L^2(\mathbb{H}^n,dm_{2(1-x)\beta,h})} = \left\|e^{-(1-z)\beta|h^{-1}\cdot|}\psi\right\|_{L^2(\mathbb{H}^n,dm_{2(1-x)\beta,h})} = 1.$$

By (10) and (5),

$$\left|F_{\varphi,\psi}(iy)\right| \leqslant C \quad\text{and}\quad \left|F_{\varphi,\psi}(1+iy)\right| \leqslant C.$$

In view of the Phragmen–Lindelöf maximal principle for the strip, we obtain

$$\left|F_{\varphi,\psi}(z)\right| \leqslant C,$$

which means

$$\left\|T^L_{e^{\frac{\pi}{2}(-y+ix)}}\right\|_{L^2(\mathbb{H}^n,dm_{2(1-x)\beta,h})\mapsto L^2(\mathbb{H}^n,dm_{2(1-x)\beta,h})} \leqslant Ce^{4b_1\beta^2 e^{-\frac{\pi}{2}y}}, \quad 0 \leqslant x \leqslant 1.$$

By the same argument,

$$\left\|T^L_{e^{\frac{\pi}{2}(-y+ix)}}\right\|_{L^2(\mathbb{H}^n,dm_{2(1+x)\beta,h})\mapsto L^2(\mathbb{H}^n,dm_{2(1+x)\beta,h})} \leqslant Ce^{4b_1\beta^2 e^{-\frac{\pi}{2}y}}, \quad -1 \leqslant x \leqslant 0.$$

An interpolation argument combining with (5) gives

$$\left\|T^L_{e^{\frac{\pi}{2}(-y+ix)}}\right\|_{L^2(\mathbb{H}^n,dm_{\beta,h})\mapsto L^2(\mathbb{H}^n,dm_{\beta,h})} \leqslant Ce^{9b_1\beta^2 e^{-\frac{\pi}{2}y}}, \quad |x| \leqslant \frac{2}{3},$$

which is equivalent to

$$\left\| T_\zeta^L \right\|_{L^2(\mathbb{H}^n, dm_{\beta,h}) \mapsto L^2(\mathbb{H}^n, dm_{\beta,h})} \leqslant C e^{b_2(\operatorname{Re}\zeta)\beta^2}, \quad |\arg\zeta| \leqslant \frac{\pi}{3}. \tag{11}$$

Let $s = \operatorname{Re}\zeta$. If $|\arg\zeta| < \frac{\pi}{4}$, then $|\arg(\zeta - \frac{s}{3})| < \frac{\pi}{3}$. Since

$$K_\zeta^L(g,h) = T_{\zeta-\frac{s}{3}}^L\big(K_{\frac{s}{3}}^L(\cdot,h)\big)(g),$$

making use of (11) and Lemma 7, we get

$$\begin{aligned}
\int_{\mathbb{H}^n} \left|K_\zeta^L(g,h)\right|^2 e^{\beta|h^{-1}g|}\,dg &\leqslant C e^{2b_2 s\beta^2} \int_{\mathbb{H}^n} \left|K_{\frac{s}{3}}^L(g,h)\right|^2 e^{\beta|h^{-1}g|}\,dg \\
&\leqslant C_N s^{-Q}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-2N} e^{2b_2 s\beta^2} \int_{\mathbb{H}^n} e^{-As^{-1}|h^{-1}g|^2} e^{\beta|h^{-1}g|}\,dg \\
&\leqslant C_N s^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-2N} e^{b_3 s\beta^2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\left|K_\zeta^L(g,h)\right| e^{\beta|h^{-1}g|} &= \left|\int_{\mathbb{H}^n} K_{\frac{\zeta}{2}}^L(g,w) K_{\frac{\zeta}{2}}^L(w,h)\,dw\right| e^{\beta|h^{-1}g|} \\
&\leqslant \left(\int_{\mathbb{H}^n} \left|K_{\frac{\zeta}{2}}^L(g,w)\right|^2 e^{2\beta|w^{-1}g|}\,dw\right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^n} \left|K_{\frac{\zeta}{2}}^L(w,h)\right|^2 e^{2\beta|h^{-1}w|}\,dw\right)^{\frac{1}{2}} \\
&\leqslant C_N s^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(g)}\right)^{-N}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N} e^{2b_3 s\beta^2}
\end{aligned}$$

because of $|K_\zeta^L(g,h)| = |K_{\bar{\zeta}}^L(h,g)|$. Plugging in $\beta = \frac{1}{4b_3 s}|h^{-1}g|$, we obtain the required estimate and complete the proof of Lemma 8. □

Let us consider the difference $E_s(g,h) = H_s(h^{-1}g) - K_s^L(g,h)$. By the perturbation theory for semigroups of operators (cf. [12, Chapter 9, formula (2.3)]),

$$\begin{aligned}
E_s(g,h) &= \int_0^s \int_{\mathbb{H}^n} H_{s-t}\big(w^{-1}g\big) V(w) K_t^L(w,h)\,dw\,dt \\
&= \int_0^{\frac{s}{2}} \int_{\mathbb{H}^n} H_{s-t}\big(w^{-1}g\big) V(w) K_t^L(w,h)\,dw\,dt \\
&\quad + \int_0^{\frac{s}{2}} \int_{\mathbb{H}^n} H_t\big(w^{-1}g\big) V(w) K_{s-t}^L(w,h)\,dw\,dt.
\end{aligned} \tag{12}$$

Lemma 9.

$$E_s(g,h) \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \cdot \min\left\{\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta}, \left(\frac{\sqrt{s}}{\rho(h)}\right)^{\delta}\right\}.$$

Proof. By the symmetry of $E_s(g,h)$, it suffices to show

$$E_s(g,h) \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \left(\frac{\sqrt{s}}{\rho(h)}\right)^{\delta}.$$

By (4), we only need to consider the case of $s < \rho(h)^2$. Write

$$I_1 := \int_0^{\frac{s}{2}} \int_{\mathbb{H}^n} H_{s-t}\big(w^{-1}g\big) V(w) K_t^L(w,h)\, dw\, dt,$$

$$I_2 := \int_0^{\frac{s}{2}} \int_{\mathbb{H}^n} H_t\big(w^{-1}g\big) V(w) K_{s-t}^L(w,h)\, dw\, dt.$$

Using Lemma 6 together with (4), we get

$$\begin{aligned}
I_1 &= \int_0^{\frac{s}{2}} \int_{|w^{-1}h| < \frac{|h^{-1}g|}{2}} H_{s-t}\big(w^{-1}g\big) V(w) K_t^L(w,h)\, dw\, dt \\
&\quad + \int_0^{\frac{s}{2}} \int_{|w^{-1}h| \geqslant \frac{|h^{-1}g|}{2}} H_{s-t}\big(w^{-1}g\big) V(w) K_t^L(w,h)\, dw\, dt \\
&\leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \int_0^{\frac{s}{2}} \int_{|w^{-1}h| < \frac{|h^{-1}g|}{2}} V(w) t^{-\frac{Q}{2}} e^{-At^{-1}|w^{-1}h|^2}\, dw\, dt \\
&\quad + Cs^{-\frac{Q}{2}} \int_0^{\frac{s}{2}} \int_{|w^{-1}h| \geqslant \frac{|h^{-1}g|}{2}} V(w) t^{-\frac{Q}{2}} e^{-At^{-1}(|w^{-1}h|^2 + |h^{-1}g|^2)}\, dw\, dt \\
&\leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \int_0^{\frac{s}{2}} \frac{1}{t} \left(\frac{\sqrt{t}}{\rho(h)}\right)^{\delta} dt \\
&= Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \left(\frac{\sqrt{s}}{\rho(h)}\right)^{\delta}.
\end{aligned}$$

The same estimate as I_1,

$$I_2 \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \left(\frac{\sqrt{s}}{\rho(g)} \right)^{\delta}.$$

By Lemma 4,

$$\frac{\sqrt{s}}{\rho(g)} \leqslant C \left(1 + \frac{|h^{-1}g|}{\sqrt{s}} \frac{\sqrt{s}}{\rho(h)} \right)^{m_0} \frac{\sqrt{s}}{\rho(h)} \leqslant C_\varepsilon e^{\varepsilon s^{-1}|h^{-1}g|^2} \frac{\sqrt{s}}{\rho(h)}$$

where $\varepsilon > 0$ is an arbitrary small constant. Hence we also have

$$I_2 \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \left(\frac{\sqrt{s}}{\rho(h)} \right)^{\delta},$$

and Lemma 9 is proved. □

Lemma 10. *Let* $0 < \delta' < \delta$. *If* $|u| \leqslant \min\{\frac{|h^{-1}g|}{4}, \rho(g)\}$, *then*

$$\left| E_s(gu,h) - E_s(g,h) \right| \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \left(\frac{|u|}{\rho(h)} \right)^{\delta'}.$$

Proof. Essentially Lemma 10 is proved by the same arguments as Lemma 9. It is enough to prove that

$$\left| E_s(gu,h) - E_s(g,h) \right| \leqslant C_\varepsilon s^{-\frac{Q}{2}} e^{\varepsilon s^{-1}|h^{-1}g|^2} \left(\frac{|u|}{\rho(h)} \right)^{\delta'}, \tag{13}$$

where $\varepsilon > 0$ is an arbitrary small constant. The case for $|u| \geqslant \rho(h)$ is trivial, so we may assume $|u| < \rho(h)$. If $s \leqslant 2|u|^2$, Lemma 9 gives the required estimate. We hence consider the case $s > 2|u|^2$ only. It is known that

$$\left| \nabla_{\mathbb{H}^n} H_s(g) \right| \leqslant Cs^{-\frac{Q+1}{2}} e^{-As^{-1}|g|^2}$$

(cf. [11]). By the mean value theorem (cf. [9]),

$$\left| H_s(gu) - H_s(g) \right| \leqslant C|u|s^{-\frac{Q+1}{2}} \tag{14}$$

and

$$\left| H_s(gu) - H_s(g) \right| \leqslant C|u|s^{-\frac{Q+1}{2}} e^{-As^{-1}|g|^2}, \quad \text{if } |u| \leqslant \frac{|g|}{2}. \tag{15}$$

According to (12), we have

$$
\begin{aligned}
\left|E_s(gu,h)-E_s(g,h)\right| &\leqslant \int_0^{\frac{s}{2}}\int_{\mathbb{H}^n}\left|H_{s-t}\left(w^{-1}gu\right)-H_{s-t}\left(w^{-1}g\right)\right|V(w)K_t^L(w,h)\,dw\,dt \\
&\quad+\int_0^{\frac{s}{2}}\int_{\mathbb{H}^n}\left|H_t\left(w^{-1}gu\right)-H_t\left(w^{-1}g\right)\right|V(w)K_{s-t}^L(w,h)\,dw\,dt \\
&= J_1+J_2.
\end{aligned}
$$

First we give the estimate for J_1. If $s<2\rho(h)^2$, using Lemma 6 and (14), we get

$$
\begin{aligned}
J_1 &\leqslant C|u|s^{-\frac{Q+1}{2}}\int_0^{\frac{s}{2}}\int_{\mathbb{H}^n}V(w)t^{-\frac{Q}{2}}e^{-At^{-1}|h^{-1}w|^2}\,dw\,dt \\
&\leqslant C|u|s^{-\frac{Q+1}{2}}\left(\frac{\sqrt{s}}{\rho(h)}\right)^{\delta}\leqslant Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\rho(h)}\right)^{\delta}.
\end{aligned}
$$

When $s\geqslant 2\rho(h)^2$, combining Lemma 6 with Lemma 7, we get

$$
\begin{aligned}
J_1 &\leqslant C|u|s^{-\frac{Q+1}{2}}\int_0^{\frac{s}{2}}\int_{\mathbb{H}^n}V(w)t^{-\frac{Q}{2}}e^{-At^{-1}|h^{-1}w|^2}\left(1+\frac{\sqrt{t}}{\rho(h)}\right)^{-N}dw\,dt \\
&\leqslant C|u|s^{-\frac{Q+1}{2}}\left(\int_0^{\rho(h)^2}\frac{1}{t}\left(\frac{\sqrt{t}}{\rho(h)}\right)^{\delta}dt+\int_{\rho(h)^2}^{\frac{s}{2}}\frac{1}{t}\left(\frac{\sqrt{t}}{\rho(h)}\right)^{l_0-N}dt\right) \\
&\leqslant C|u|s^{-\frac{Q+1}{2}}\leqslant Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\rho(h)}\right)^{\delta},
\end{aligned}
$$

where N is chosen large enough satisfying $N>l_0$.

To estimate J_2, we use Lemma 7 and write

$$
\begin{aligned}
J_2 &\leqslant Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_0^{\frac{s}{2}}\int_{\mathbb{H}^n}\left|H_t\left(w^{-1}gu\right)-H_t\left(w^{-1}g\right)\right|V(w)\,dw\,dt \\
&= Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_0^{|u|^2}\int_{\mathbb{H}^n}\left|H_t\left(w^{-1}gu\right)-H_t\left(w^{-1}g\right)\right|V(w)\,dw\,dt \\
&\quad+Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_{|u|^2}^{\frac{s}{2}}\int_{|w^{-1}g|<2|u|}\left|H_t\left(w^{-1}gu\right)-H_t\left(w^{-1}g\right)\right|V(w)\,dw\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+ Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_{|u|^2}^{\frac{s}{2}}\int_{|w^{-1}g|\geqslant 2|u|}\left|H_t\left(w^{-1}gu\right)-H_t\left(w^{-1}g\right)\right|V(w)\,dw\,dt \\
&= J_{2,1}+J_{2,2}+J_{2,3}.
\end{aligned}
$$

Note that $\rho(gu)\sim\rho(g)$ as $|u|\leqslant\rho(g)$. We have

$$
\begin{aligned}
J_{2,1} &\leqslant Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_0^{|u|^2}\int_{\mathbb{H}^n} t^{-\frac{Q}{2}}\left(e^{-At^{-1}|w^{-1}gu|^2}+e^{-At^{-1}|w^{-1}g|^2}\right)V(w)\,dw\,dt \\
&\leqslant Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_0^{|u|^2}\frac{1}{t}\left(\frac{\sqrt{t}}{\rho(g)}\right)^{\delta}dt \\
&= Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\rho(h)}\right)^{\delta}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{\rho(h)}{\rho(g)}\right)^{\delta}.
\end{aligned}
$$

Using Lemmas 1–3 and (14), we get

$$
\begin{aligned}
J_{2,2} &\leqslant Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_{|u|^2}^{\frac{s}{2}}\int_{|w^{-1}g|<2|u|}|u|t^{-\frac{Q+1}{2}}V(w)\,dw\,dt \\
&\leqslant Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_{|u|^2}^{\frac{s}{2}}|u|^{Q-1}t^{-\frac{Q+1}{2}}\left(\frac{|u|}{\rho(g)}\right)^{\delta}dt \\
&\leqslant Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\rho(h)}\right)^{\delta}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{\rho(h)}{\rho(g)}\right)^{\delta}.
\end{aligned}
$$

Suppose $s\leqslant 2\rho(g)^2$. Using Lemma 6 together with (15), we get

$$
\begin{aligned}
J_{2,3} &\leqslant Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_{|u|^2}^{\frac{s}{2}}\int_{|w^{-1}g|\geqslant 2|u|}|u|t^{-\frac{Q+1}{2}}e^{-At^{-1}|w^{-1}g|^2}V(w)\,dw\,dt \\
&\leqslant Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int_{|u|^2}^{\frac{s}{2}}\frac{1}{t}\left(\frac{|u|}{\rho(g)}\right)^{\delta}dt \\
&\leqslant Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\rho(h)}\right)^{\delta}\left(1+\left|\log\frac{|u|}{\rho(h)}\right|\right)\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{\rho(h)}{\rho(g)}\right)^{\delta}\left(1+\left|\log\frac{\rho(h)}{\rho(g)}\right|\right).
\end{aligned}
$$

If $s>2\rho(g)^2$, then

$$
\begin{aligned}
J_{2,3} &\leqslant Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int\limits_{|u|^2}^{\rho(g)^2}\int\limits_{|w^{-1}g|\geqslant 2|u|}\left|H_t\left(w^{-1}gu\right)-H_t\left(w^{-1}g\right)\right|V(w)\,dw\,dt\\
&\quad+Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int\limits_{\rho(g)^2}^{\frac{s}{2}}\int\limits_{|w^{-1}g|\geqslant 2|u|}|u|t^{-\frac{Q+1}{2}}e^{-At^{-1}|w^{-1}g|^2}V(w)\,dw\,dt\\
&\leqslant Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\rho(h)}\right)^{\delta}\left(1+\left|\log\frac{|u|}{\rho(h)}\right|\right)\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{\rho(h)}{\rho(g)}\right)^{\delta}\left(1+\left|\log\frac{\rho(h)}{\rho(g)}\right|\right)\\
&\quad+Cs^{-\frac{Q}{2}}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\int\limits_{\rho(g)^2}^{\frac{s}{2}}\frac{|u|}{\sqrt{t}}\frac{1}{t}\left(\frac{\sqrt{t}}{\rho(g)}\right)^{l_0}dt\\
&\leqslant Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\rho(h)}\right)^{\delta}\left(1+\left|\log\frac{|u|}{\rho(h)}\right|\right)\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{\rho(h)}{\rho(g)}\right)^{\delta}\left(1+\left|\log\frac{\rho(h)}{\rho(g)}\right|\right)\\
&\quad+Cs^{-\frac{Q}{2}}\frac{|u|}{\rho(h)}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N+l_0-1}\left(\frac{\rho(h)}{\rho(g)}\right)^{l_0},
\end{aligned}
$$

where we use the estimate already obtained for $s=2\rho(g)^2$ and Lemma 6 for $t\geqslant\rho(g)^2$ in the second inequality.

By Lemma 4,

$$
\frac{\rho(h)}{\rho(g)}\leqslant C\left(1+\frac{|h^{-1}g|}{\sqrt{s}}\frac{\sqrt{s}}{\rho(h)}\right)^{m_0}\leqslant C_\varepsilon e^{\varepsilon s^{-1}|h^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{m_0},
$$

where $\varepsilon>0$ is an arbitrary small constant. Choosing N large enough in the estimates of $J_{2,1}$, $J_{2,2}$ and $J_{2,3}$, we obtain (13) and hence Lemma 10 is proved. $\square$

Next lemma establishes the Lipschitz regularity of the kernel K_s^L.

Lemma 11. *Let $0<\delta'<\delta$ and $|u|\leqslant\sqrt{s}$. For any $N>0$, there exists a constant $C_N>0$ such that*

$$
\left|K_s^L(gu,h)-K_s^L(g,h)\right|\leqslant C_N s^{-\frac{Q}{2}}e^{-As^{-1}|h^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'}. \tag{16}
$$

Proof. We assume first that $|u|\leqslant\frac{|h^{-1}g|}{4}$. Since $|u|\leqslant\sqrt{s}$, by Lemma 4,

$$
\frac{\sqrt{s}}{\rho(g)}\leqslant C\left(1+\frac{|u|}{\rho(g)}\right)^{\frac{m_0}{m_0+1}}\frac{\sqrt{s}}{\rho(gu)}\leqslant C\left(1+\frac{\sqrt{s}}{\rho(g)}\right)^{\frac{m_0}{m_0+1}}\frac{\sqrt{s}}{\rho(gu)},
$$

and hence

$$
\left(1+\frac{\sqrt{s}}{\rho(gu)}\right)^{-1}\leqslant C\left(1+\frac{\sqrt{s}}{\rho(g)}\right)^{-\frac{1}{m_0+1}}.
$$

By Lemma 7, for any $N > 0$ there exists a constant $C_N > 0$ such that

$$\left|K_s^L(gu,h) - K_s^L(g,h)\right| \leqslant C_N s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(1 + \frac{\sqrt{s}}{\rho(g)} + \frac{\sqrt{s}}{\rho(h)}\right)^{-N}. \tag{17}$$

If $|u| \geqslant \rho(g)$, then (16) follows from (17). If $|u| < \rho(g)$, by Lemma 10, for $0 < \delta'' < \delta$,

$$\left|E_s(gu,h) - E_s(g,h)\right| \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta''}\left(\frac{\sqrt{s}}{\rho(h)}\right)^{\delta''}.$$

In view of (15), we have

$$\left|K_s^L(gu,h) - K_s^L(g,h)\right| \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta''}\left(1 + \frac{\sqrt{s}}{\rho(h)}\right)^{\delta''}. \tag{18}$$

Then (16) follows from (17) and (18).

Next we consider the case $|u| > \frac{|h^{-1}g|}{4}$. Using the semigroup property of T_s^L, we have

$$\begin{aligned}
\left|K_s^L(gu,h) - K_s^L(g,h)\right| &\leqslant \int_{\mathbb{H}^n} \left|K_{\frac{s}{2}}^L(gu,w) - K_{\frac{s}{2}}^L(g,w)\right| K_{\frac{s}{2}}^L(w,h)\,dw \\
&= \int_{|w^{-1}g|<4|u|} \left|K_{\frac{s}{2}}^L(gu,w) - K_{\frac{s}{2}}^L(g,w)\right| K_{\frac{s}{2}}^L(w,h)\,dw \\
&\quad + \int_{|w^{-1}g|\geqslant 4|u|} \left|K_{\frac{s}{2}}^L(gu,w) - K_{\frac{s}{2}}^L(g,w)\right| K_{\frac{s}{2}}^L(w,h)\,dw \\
&= I_1 + I_2.
\end{aligned}$$

By Lemma 7,

$$I_1 \leqslant C_N s^{-Q}\left(1 + \frac{\sqrt{s}}{\rho(h)}\right)^{-N}|u|^Q \leqslant C_N s^{-\frac{Q}{2}}\left(1 + \frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'}.$$

For $|u| \leqslant \frac{|w^{-1}g|}{4}$, we just proved

$$\left|K_{\frac{s}{2}}^L(gu,w) - K_{\frac{s}{2}}^L(g,w)\right| \leqslant Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'}$$

and hence

$$I_2 \leqslant Cs^{-\frac{Q}{2}}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'}\int_{|w^{-1}g|\geqslant 4|u|} K_{\frac{s}{2}}^L(w,h)\,dw \leqslant C_N s^{-\frac{Q}{2}}\left(1 + \frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'}.$$

Note that $|h^{-1}g| < 4\sqrt{s}$. We have

$$e^{-s^{-1}|h^{-1}g|^2} \geqslant C$$

and, by Lemma 4,

$$\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-1} \leqslant C\left(1+\frac{\sqrt{s}}{\rho(g)}\right)^{-\frac{1}{m_0+1}}.$$

Therefore, for any $N>0$, there exists a constant $C_N>0$ such that

$$I_1+I_2 \leqslant C_N s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'}.$$

The proof of Lemma 11 is completed. □

Recall the operator Q_s^L defined in Section 2 by $Q_s^L\varphi(g) = s\frac{d}{ds}T_s^L\varphi(g)$. Let $Q_s^L(g,h)$ denote the kernel of Q_s^L. Then

$$Q_s^L(g,h) = s\partial_s K_s^L(g,h). \tag{19}$$

Lemma 12. *$Q_s^L(g,h)$ satisfies the following estimates:*

(a) *For any $N>0$, there exists a constant $C_N>0$ such that*

$$\left|Q_s^L(g,h)\right| \leqslant C_N s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}.$$

(b) *Let $0<\delta'<\delta$ and $|u|\leqslant\sqrt{s}$. For any $N>0$, there exists a constant $C_N>0$ such that*

$$\left|Q_s^L(gu,h)-Q_s^L(g,h)\right| \leqslant C_N s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'}.$$

(c) *For any $N>0$, there exists a constant $C_N>0$ such that*

$$\left|\int_{\mathbb{H}^n} Q_s^L(g,h)\,dh\right| \leqslant C_N\left(1+\frac{\sqrt{s}}{\rho(g)}\right)^{-N}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta}.$$

Proof. By the Cauchy integral formula and (19) combined with Lemma 8, we get

$$\begin{aligned}\left|Q_s^L(g,h)\right| &= \left|\frac{1}{2\pi i}\int_{|\zeta-s|=\frac{s}{2}} \frac{sK_\zeta^L(g,h)}{(\zeta-s)^2}\,d\zeta\right| \\ &\leqslant C_N s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\end{aligned}$$

that proves (a).

In view of the semigroup property of T_s^L and making use of Lemma 11 with the part (a) already proved, we obtain

$$\begin{aligned}
&\left|Q_s^L(gu,h)-Q_s^L(g,h)\right| \\
&\quad=\left|2\int_{\mathbb{H}^n}\left(K_{\frac{s}{2}}^L(gu,w)-K_{\frac{s}{2}}^L(g,w)\right)Q_{\frac{s}{2}}^L(w,h)\,dw\right| \\
&\quad\leqslant C_N\int_{\mathbb{H}^n}s^{-Q}e^{-As^{-1}|w^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}\right)^{-N}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'}e^{-As^{-1}|h^{-1}w|^2}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}dw \\
&\quad\leqslant C_N s^{-\frac{Q}{2}}e^{-As^{-1}|h^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}\left(\frac{|u|}{\sqrt{s}}\right)^{\delta'},
\end{aligned}$$

and (b) is established.

Note that

$$\left|\int_{\mathbb{H}^n}Q_s^L(g,h)\,dh\right|=s\left|T_s^L L\mathbf{1}(g)\right|=s\int_{\mathbb{H}^n}K_s^L(g,h)V(h)\,dh,$$

where $\mathbf{1}$ denotes the constant function of value 1. Then (c) is easily deduced from Lemmas 6 and 7. $\square$

Remark 4. It is easy to see that we can replace the condition $|u|\leqslant\sqrt{s}$ by $|u|\leqslant\frac{|h^{-1}g|}{2}$ in Lemmas 11 and 12(b).

4. Duality of $H_L^1(\mathbb{H}^n)$ and $BMO_L(\mathbb{H}^n)$

In this section we give the proof of Theorem 1. The proof of part (a) is standard. For any $H_L^{1,\infty}$-atom a, it follows directly from the definition of $BMO_L(\mathbb{H}^n)$ that

$$\left|\mathcal{L}_\varphi(a)\right|=\left|\int_{\mathbb{H}^n}a(g)\varphi(g)\,dg\right|\leqslant C\|\varphi\|_{BMO_L}.$$

Next let $\varphi\in L^\infty(\mathbb{H}^n)\subset BMO_L(\mathbb{H}^n)$ and $f\in L_c^\infty$. Write $f=\sum_j\lambda_j a_j$ in $L^1(\mathbb{H}^n)$, where the a_j's are $H_L^{1,\infty}$-atoms and $\sum_j|\lambda_j|\sim\|f\|_{H_L^1}$. Then

$$\begin{aligned}
\left|\mathcal{L}_\varphi(f)\right|&=\left|\sum_j\lambda_j\int_{\mathbb{H}^n}a_j(g)\varphi(g)\,dg\right|\leqslant C\|\varphi\|_{BMO_L}\sum_j|\lambda_j| \\
&\leqslant C\|\varphi\|_{BMO_L}\|f\|_{H_L^1}.
\end{aligned}$$

We might as well assume that $\varphi\in BMO_L(\mathbb{H}^n)$ is real-valued. Set

$$\varphi_k(g)=\begin{cases}k, & \text{if } \varphi(g)>k,\\ \varphi(g), & \text{if } -k\leqslant\varphi(g)\leqslant k,\\ -k, & \text{if } \varphi(g)<-k.\end{cases}$$

It is easy to see that $\|\varphi_k\|_{BMO_L}\leqslant C\|\varphi\|_{BMO_L}$. Then, for $f\in L_c^\infty$, we have

$$\left|\mathcal{L}_\varphi(f)\right|=\lim_{k\to\infty}\left|\int_{\mathbb{H}^n} f(g)\varphi_k(g)\,dg\right|\leqslant C\|\varphi\|_{BMO_L}\|f\|_{H_L^1},$$

which proves assertion (a) of Theorem 1.

Now we prove assertion (b) of Theorem 1. Let L_c^2 denote the space of all square integrable functions with compact support. Then L_c^2 is exactly the space of finite linear combinations of $H_L^{1,2}$-atoms. By Proposition 1, L_c^2 is a dense subspace of $H_L^1(\mathbb{H}^n)$. Moreover, for every large ball $B=B(g,r)$ with $r\geqslant\rho(g)$, if f is a square integrable function supported in B, then

$$\|f\|_{H_L^1}\leqslant C|B|^{\frac{1}{2}}\|f\|_{L^2(B)}. \tag{20}$$

Given $\mathcal{L}\in(H_L^1(\mathbb{H}^n))^*$, set $B_R=B(0,R)$ with $R>\rho(0)$. By the Riesz representation theorem, there exists a unique function $\varphi_R\in L^2(B_R)$ such that

$$\mathcal{L}(f)=\int_{B_R} f(g)\varphi_R(g)\,dg\quad\text{for } f\in L^2(B_R).$$

It is obvious that the restriction of φ_{R_2} on B_{R_1} coincides with φ_{R_1} if $R_1<R_2$. Thus we have a unique locally square integrable function φ on $\mathbb{H}^n$ such that

$$\mathcal{L}(f)=\int_{\mathbb{H}^n} f(g)\varphi(g)\,dg$$

for all square integrable functions f with compact support. That is $\mathcal{L}=\mathcal{L}_\varphi$. It remains to show

$$\|\varphi\|_{BMO_L}\leqslant C\|\mathcal{L}\|.$$

Let $B=B(g,r)$. If $r\geqslant\rho(g)$, then, by (20),

$$\|\varphi\|_{L^2(B)}\leqslant C|B|^{\frac{1}{2}}\|\mathcal{L}\|,$$

and hence

$$\frac{1}{|B(g,r)|}\int_{B(g,r)}\left|\varphi(h)\right|dh\leqslant\left(\frac{1}{|B(g,r)|}\int_{B(g,r)}\left|\varphi(h)\right|^2dh\right)^{\frac{1}{2}}\leqslant C\|\mathcal{L}\|.$$

On the other hand, the usual Hardy space $H^1(\mathbb{H}^n)$ is contained in $H_L^1(\mathbb{H}^n)$ and $\|f\|_{H_L^1}\leqslant C\|f\|_{H^1}$. It follows that $\mathcal{L}|_{H^1}\in(H^1(\mathbb{H}^n))^*$. This means $\varphi\in BMO(\mathbb{H}^n)$ and

$$\frac{1}{|B(g,r)|}\int\limits_{B(g,r)} \left|\varphi(h)-\varphi(B)\right| dh \leqslant \|\varphi\|_{BMO} \leqslant C\|\mathcal{L}\|$$

(cf. [9]). The proof of Theorem 1 is completed.

Remark 5. Let $1 < p < \infty$. If $\varphi \in BMO(\mathbb{H}^n)$, then for any ball $B = B(g,r)$,

$$\left(\frac{1}{|B(g,r)|}\int\limits_{B(g,r)} \left|\varphi(h)-\varphi(B)\right|^p dh\right)^{\frac{1}{p}} \leqslant C\|\varphi\|_{BMO}$$

(cf. [9]). In the above we have proved that if $\varphi \in BMO_L(\mathbb{H}^n)$, then

$$\left(\frac{1}{|B(g,r)|}\int\limits_{B(g,r)} \left|\varphi(h)-\varphi(B,V)\right|^2 dh\right)^{\frac{1}{2}} \leqslant C\|\varphi\|_{BMO_L}.$$

In the same way,

$$\left(\frac{1}{|B(g,r)|}\int\limits_{B(g,r)} \left|\varphi(h)-\varphi(B,V)\right|^p dh\right)^{\frac{1}{p}} \leqslant C\|\varphi\|_{BMO_L}.$$

The John–Nirenberg inequality also holds for $\varphi \in BMO_L(\mathbb{H}^n)$.

5. Riesz transforms and the Fefferman–Stein decomposition

In this section we prove Theorem 2. First we prove assertion (a). Let $\varphi \in BMO_L(\mathbb{H}^n)$ and $B = B(g_0, r)$. Assume $r \geqslant \rho(g_0)$. We set

$$\varphi = \varphi\chi_{B^*} + \varphi\chi_{(B^*)^c} = \varphi_1 + \varphi_2,$$

where $B^* = B(g_0, 2r)$ and χ_S denotes the characteristic function of a set S. Since $\widetilde{R}_j^L$ are bounded on $L^2(\mathbb{H}^n)$, by Remark 5, we have

$$\begin{aligned}
\frac{1}{|B|}\int\limits_{B} \left|\widetilde{R}_j^L\varphi_1(g)\right| dg &\leqslant \left(\frac{1}{|B|}\int\limits_{B} \left|\widetilde{R}_j^L\varphi_1(g)\right|^2 dg\right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{C}{|B|}\int\limits_{B^*} \left|\varphi_1(g)\right|^2 dg\right)^{\frac{1}{2}} \\
&= \left(\frac{C}{|B^*|}\int\limits_{B^*} \left|\varphi(g)\right|^2 dg\right)^{\frac{1}{2}} \\
&\leqslant C\|\varphi\|_{BMO_L}. \qquad (21)
\end{aligned}$$

Let $g \in B(g_0, r)$. If $\rho(g) > r$, by Lemma 4, $\rho(g) \sim \rho(g_0) \leqslant r$. Thus we always have $\rho(g) \leqslant Cr$. Note that the kernels of $\widetilde{R}_j^L$ are given by $\widetilde{R}_j^L(g,h) = -R_j^L(h,g)$ where $R_j^L(g,h)$ are the kernels of R_j^L. For $k \geqslant 1$, we have the estimates

$$\left(\int_{2^{k-1}r \leqslant |h^{-1}g| < 2^k r} \left|\widetilde{R}_j^L(g,h)\right|^{p_0} dh \right)^{\frac{1}{p_0}} \leqslant C2^{-k}\left(2^k r\right)^{-\frac{Q}{p_0'}}$$

for some $p_0 > Q$ (cf. [14, proof of Lemma 8]). It follows that

$$\left(\int_{2^{k-1}r \leqslant |h^{-1}g| < 2^k r} \left|\widetilde{R}_j^L(g,h)\right|^{2} dh \right)^{\frac{1}{2}} \leqslant C2^{-k}\left(2^k r\right)^{-\frac{Q}{2}}.$$

Then we get

$$\begin{aligned}
\left|\widetilde{R}_j^L \varphi_2(g)\right| &\leqslant \int_{(B^*)^c} \left|\widetilde{R}_j^L(g,h)\right|\left|\varphi(h)\right| dh \\
&\leqslant \sum_{k=1}^{\infty} \left(\int_{2^{k-1}r \leqslant |h^{-1}g| < 2^k r} \left|\widetilde{R}_j^L(g,h)\right|^2 dh \right)^{\frac{1}{2}} \left(\int_{|h^{-1}g| < 2^k r} \left|\varphi(h)\right|^2 dh \right)^{\frac{1}{2}} \\
&\leqslant C \sum_{k=1}^{\infty} 2^{-k} \|\varphi\|_{BMO_L} = C\|\varphi\|_{BMO_L}.
\end{aligned}$$

Hence

$$\frac{1}{|B|} \int_B \left|\widetilde{R}_j^L \varphi_2(g)\right| dg \leqslant C\|\varphi\|_{BMO_L}. \tag{22}$$

The above argument also shows that $\widetilde{R}_j^L$ are well defined on $BMO_L(\mathbb{H}^n)$ without the ambiguity of an additive constant.

When $r < \rho(g_0)$, we set

$$\varphi = \varphi\chi_{B^\sharp} + \varphi\chi_{(B^\sharp)^c} = \varphi_1' + \varphi_2',$$

where $B^\sharp = B(g_0, 2\rho(g_0))$. Note that $\rho(g) \sim \rho(g_0)$ for any $g \in B(g_0, r)$. By the same argument as (22), we have

$$\frac{1}{|B|} \int_B \left|\widetilde{R}_j^L \varphi_2'(g)\right| dg \leqslant C\|\varphi\|_{BMO_L}. \tag{23}$$

We claim that there exists a constant A_0 such that

$$\frac{1}{|B|} \int_B \left|\widetilde{R}_j^L \varphi_1'(g) - A_0\right| dg \leqslant C\|\varphi\|_{BMO_L}. \tag{24}$$

Then assertion (a) follows directly from (21)–(24).

To prove (24), we write

$$\frac{1}{|B|}\int_B \left|\widetilde{R}_j^L\varphi_1'(g)-A_0\right|dg \leqslant \frac{1}{|B|}\int_B \left(\left|\widetilde{R}_j^L\varphi_1'(g)-\widetilde{R}_j\varphi_1'(g)\right|+\left|\widetilde{R}_j\varphi_1'(g)-A_0\right|\right)dg,$$

where $\widetilde{R}_j(g)$ is the convolution kernels of $\widetilde{R}_j=(-\Delta_{\mathbb{H}^n})^{-\frac{1}{2}}X_j$. Let $g\in B(g_0,r)$ and $B_{g,k}=B(g,2^{2-k}\rho(g_0))$, $k=0,1,\ldots$. Because $\rho(g)\sim\rho(g_0)$, $|\varphi(B_{g,0})|\leqslant C\|\varphi\|_{BMO_L}$. Since

$$\left|\varphi(B_{g,k})-\varphi(B_{g,k-1})\right|\leqslant C\|\varphi\|_{BMO},$$

we have

$$\left|\varphi(B_{g,k})\right|\leqslant\left|\varphi(B_{g,0})\right|+\sum_{j=1}^{k}\left|\varphi(B_{g,j})-\varphi(B_{g,j-1})\right|\leqslant C(k+1)\|\varphi\|_{BMO_L}.$$

It follows that

$$\begin{aligned}\left(\int_{B_{g,k}}\left|\varphi(h)\right|^2dh\right)^{\frac{1}{2}}&\leqslant\left(\int_{B_{g,k}}\left|\varphi(h)-\varphi(B_{g,k})\right|^2dh\right)^{\frac{1}{2}}+|B_{g,k}|^{\frac{1}{2}}\left|\varphi(B_{g,k})\right|\\&\leqslant C(k+1)|B_{g,k}|^{\frac{1}{2}}\|\varphi\|_{BMO_L}.\end{aligned}$$

For $k\geqslant 0$, we have the estimates

$$\left(\int_{B_{g,k}\setminus B_{g,k+1}}\left|\widetilde{R}_j^L(g,h)-\widetilde{R}_j\left(h^{-1}g\right)\right|^{p_0}dh\right)^{\frac{1}{p_0}}\leqslant C2^{-k(1-\frac{Q}{p_0})}\left(2^{-k}\rho(g_0)\right)^{-\frac{Q}{p_0'}}$$

for some $p_0>Q$ (cf. [14, Proof of Lemma 9]). It follows that

$$\left(\int_{B_{g,k}\setminus B_{g,k+1}}\left|\widetilde{R}_j^L(g,h)-\widetilde{R}_j\left(h^{-1}g\right)\right|^{2}dh\right)^{\frac{1}{2}}\leqslant C2^{-k(1-\frac{Q}{p_0})}\left(2^{-k}\rho(g_0)\right)^{-\frac{Q}{2}}.$$

Then we get

$$\begin{aligned}&\left|\widetilde{R}_j^L\varphi_1'(g)-\widetilde{R}_j\varphi_1'(g)\right|\\&\quad\leqslant\sum_{k=0}^{\infty}\int_{B_{g,k}\setminus B_{g,k+1}}\left|\widetilde{R}_j^L(g,h)-\widetilde{R}_j\left(h^{-1}g\right)\right|\left|\varphi(h)\right|dh\end{aligned}$$

$$\leqslant \sum_{k=0}^{\infty}\biggl(\int\limits_{B_{g,k}\setminus B_{g,k+1}} \bigl|\widetilde{R}_j^L(g,h)-\widetilde{R}_j\bigl(h^{-1}g\bigr)\bigr|^2\,dh\biggr)^{\frac{1}{2}}\biggl(\int\limits_{B_{g,k}} \bigl|\varphi(h)\bigr|^2\,dh\biggr)^{\frac{1}{2}}$$

$$\leqslant C\sum_{k=0}^{\infty}(k+1)2^{-k(1-\frac{Q}{p_0})}\|\varphi\|_{BMO_L}\leqslant C\|\varphi\|_{BMO_L}.$$

It remains to show

$$\frac{1}{|B|}\int\limits_B \bigl|\widetilde{R}_j\varphi_1'(g)-A_0\bigr|\,dg\leqslant C\|\varphi\|_{BMO_L}. \tag{25}$$

Let $B_k^\sharp=B(g_0,2^{1-k}\rho(g_0))$, $k=0,1,\ldots,k_0$, where k_0 satisfies $2^{-k_0-1}\rho(g_0)\leqslant r<2^{-k_0}\rho(g_0)$. Since $\widetilde{R}_j\mathbf{1}=0$, $\widetilde{R}_j\varphi_1'=\widetilde{R}_j(\varphi_1'-\varphi(B_{k_0}^\sharp))$. Set

$$\begin{aligned}\varphi_1'-\varphi\bigl(B_{k_0}^\sharp\bigr)&=\bigl(\varphi-\varphi\bigl(B_{k_0}^\sharp\bigr)\bigr)\chi_{B_{k_0}^\sharp}+\bigl(\varphi-\varphi\bigl(B_{k_0}^\sharp\bigr)\bigr)\chi_{B_0^\sharp\setminus B_{k_0}^\sharp}-\varphi\bigl(B_{k_0}^\sharp\bigr)\chi_{(B_0^\sharp)^c}\\&=\varphi_{1,1}+\varphi_{1,2}+\varphi_{1,3}.\end{aligned}$$

Because $\widetilde{R}_j$ are bounded on $L^2(\mathbb{H}^n)$, we have

$$\begin{aligned}\frac{1}{|B|}\int\limits_B \bigl|\widetilde{R}_j\varphi_{1,1}(g)\bigr|\,dg&\leqslant\biggl(\frac{1}{|B|}\int\limits_B \bigl|\widetilde{R}_j\varphi_{1,1}(g)\bigr|^2\,dg\biggr)^{\frac{1}{2}}\\&\leqslant\biggl(\frac{C}{|B_{k_0}^\sharp|}\int\limits_{B_{k_0}^\sharp}\bigl|\varphi(g)-\varphi\bigl(B_{k_0}^\sharp\bigr)\bigr|^2\,dg\biggr)^{\frac{1}{2}}\\&\leqslant C\|\varphi\|_{BMO}.\end{aligned}$$

It is well known that $\widetilde{R}_j(g)$ are Calderón–Zygmund kernels satisfying

$$\bigl|\widetilde{R}_j(gu)-\widetilde{R}_j(g)\bigr|\leqslant C\frac{|u|}{|g|^{Q+1}}\quad\text{for } |u|\leqslant\frac{|g|}{2}.$$

Then we have

$$\begin{aligned}&\frac{1}{|B|}\int\limits_B \bigl|\widetilde{R}_j\varphi_{1,2}(g)-\widetilde{R}_j\varphi_{1,2}(g_0)\bigr|\,dg\\&\quad\leqslant\frac{1}{|B|}\int\limits_{g\in B}\sum_{k=0}^{k_0-1}\int\limits_{h\in B_k^\sharp\setminus B_{k+1}^\sharp}\bigl|\widetilde{R}_j(h^{-1}g)-\widetilde{R}_j(h^{-1}g_0)\bigr|\bigl|\varphi(h)-\varphi\bigl(B_{k_0}^\sharp\bigr)\bigr|\,dh\,dg\end{aligned}$$

$$\leqslant \frac{C}{|B|} \int\limits_{g \in B} \sum_{k=0}^{k_0-1} \frac{2^{k-k_0}}{|B_k^\sharp|} \int\limits_{h \in B_k^\sharp} \left(\left|\varphi(h) - \varphi\left(B_k^\sharp\right)\right| + \left|\varphi\left(B_k^\sharp\right) - \varphi\left(B_{k_0}^\sharp\right)\right| \right) dh\, dg$$

$$\leqslant C \sum_{k=0}^{k_0-1} (k_0 - k + 1) 2^{k-k_0} \|\varphi\|_{BMO} \leqslant C \|\varphi\|_{BMO}.$$

Note that $\widetilde{R}_j \mathbf{1} = 0$ implies $\widetilde{R}_j(\chi_E) = -\widetilde{R}_j(\chi_{E^c})$, and hence $\widetilde{R}_j \varphi_{1,3}(g_0)$ is well-defined. A similar argument yields

$$\frac{1}{|B|} \int\limits_B \left| \widetilde{R}_j \varphi_{1,3}(g) - \widetilde{R}_j \varphi_{1,3}(g_0) \right| dg$$

$$\leqslant \frac{|\varphi(B_{k_0}^\sharp)|}{|B|} \int\limits_{g \in B} \int\limits_{h \in (B_0^\sharp)^c} \left| \widetilde{R}_j\left(h^{-1} g\right) - \widetilde{R}_j\left(h^{-1} g_0\right) \right| dh\, dg$$

$$\leqslant C (k_0 + 1) 2^{-k_0} \|\varphi\|_{BMO_L} \leqslant C \|\varphi\|_{BMO_L}.$$

Thus (25) holds, and assertion (a) is proved.

The "if" part of assertion (b) is a direct consequence of assertion (a). We sketch the proof of "only if" part of the assertion (b). The argument is the same as the one in [8]. Let **B** be the Banach space of the direct sum of $2n+1$ copies of $L^1(\mathbb{H}^n)$. By Proposition 2, $H_L^1(\mathbb{H}^n)$ can be identified with a closed subspace of **B** by identifying f with $(f, R_1^L f, \ldots, R_{2n}^L f)$. For $\varphi \in BMO_L(\mathbb{H}^n)$, let $\mathcal{L} = \mathcal{L}_\varphi \in (H_L^1(\mathbb{H}^n))^*$. By the Hahn–Banach theorem, $\mathcal{L}$ extends to a continuous linear functional on **B**. Thus there exist $\varphi_0, \varphi_1, \ldots, \varphi_{2n} \in L^\infty(\mathbb{H}^n)$ such that

$$\mathcal{L}(f) = \int\limits_{\mathbb{H}^n} f(g) \varphi_0(g)\, dg + \sum_{j=1}^{2n} \int\limits_{\mathbb{H}^n} R_j^L f(g) \varphi_j(g)\, dg$$

$$= \int\limits_{\mathbb{H}^n} f(g) \left(\varphi_0(g) - \sum_{j=1}^{2n} \widetilde{R}_j^L \varphi_j(g) \right) dg.$$

This proves assertion (b).

6. Square functions

In this section we deal with the Littlewood–Paley function s_Q^L and the Lusin area integral S_Q^L related to L. We divide the proof of Theorem 4 into several lemmas.

Lemma 13. *The operators s_Q^L and S_Q^L are isometries on $L^2(\mathbb{H}^n)$ up to constant factors. Exactly,*

$$\left\| s_Q^L f \right\|_{L^2} = \frac{1}{2} \|f\|_{L^2} \quad \textit{and} \quad \left\| S_Q^L f \right\|_{L^2} = \frac{\sqrt{c_n}}{2} \|f\|_{L^2},$$

where c_n is the volume of the unit ball.

Proof. Since the spectrum of L is contained in $[0,\infty)$, we follow the argument in [17, p. 74] and get $\|s_Q^L f\|_{L^2} = \frac{1}{2}\|f\|_{L^2}$. As a consequence, we have

$$\begin{aligned}\|S_Q^L f\|_{L^2}^2 &= \int\limits_{\mathbb{H}^n}\int\limits_{\mathbf{U}^n} \left|Q_s^L f(h)\right|^2 \chi_{\Gamma(g)}(h,s)\,\frac{dh\,ds}{s^{\frac{Q}{2}+1}}\,dg \\ &= c_n \int\limits_{\mathbf{U}^n} \left|Q_s^L f(h)\right|^2 \frac{dh\,ds}{s} \\ &= c_n \left\|s_Q^L f\right\|_{L^2}^2 = \frac{c_n}{4}\|f\|_{L^2}^2\end{aligned}$$

and the L^2 equality for S_Q^L is proved. $\square$

Lemma 14. *Suppose a is an $H_L^{1,\infty}$-atom. Then there is an absolute constant C such that*

$$\left\|s_Q^L a\right\|_{L^1} \leqslant C \quad \textit{and} \quad \left\|S_Q^L a\right\|_{L^1} \leqslant C.$$

Proof. The proofs for s_Q^L and S_Q^L are essentially the same, so we give the proof for S_Q^L only. If a is an $H_L^{1,\infty}$-atom supported on the ball $B(g_0,r)$, then

$$\left\|S_Q^L a\right\|_{L^1(B(g_0,4r))} \leqslant \left|B(g_0,4r)\right|^{\frac{1}{2}} \left\|S_Q^L a\right\|_{L^2} \leqslant C\left|B(g_0,r)\right|^{\frac{1}{2}} \|a\|_{L^2} \leqslant C. \tag{26}$$

Let $g \notin B(g_0,4r)$. If $r < \rho(g_0)$, then a satisfies the vanishing condition, and hence

$$\begin{aligned}S_Q^L a(g)^2 &\leqslant \int\limits_{\Gamma(g)} \left(\int\limits_{B(g_0,r)} \left|Q_s^L(h,w) - Q_s^L(h,g_0)\right|\left|a(w)\right| dw \right)^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}} \\ &\leqslant \int\limits_0^{\frac{1}{4}|g^{-1}g_0|^2} \int\limits_{|g^{-1}h|<\sqrt{s}} \left(\int\limits_{B(g_0,r)} \left|Q_s^L(h,w) - Q_s^L(h,g_0)\right|\left|a(w)\right| dw \right)^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}} \\ &\quad + \int\limits_{\frac{1}{4}|g^{-1}g_0|^2}^{\infty} \int\limits_{|g^{-1}h|<\sqrt{s}} \left(\int\limits_{B(g_0,r)} \left|Q_s^L(h,w) - Q_s^L(h,g_0)\right|\left|a(w)\right| dw \right)^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}} \\ &= I_1 + I_2.\end{aligned}$$

For any $w \in B(g_0,r)$ and $g \notin B(g_0,4r)$, when $|g^{-1}h| < \sqrt{s} \leqslant \frac{1}{2}|g^{-1}g_0|$, we have $|w^{-1}g_0| < r \leqslant \frac{1}{2}|h^{-1}g_0|$ and $|h^{-1}w| \sim |h^{-1}g_0| \sim |g^{-1}g_0|$. By Lemma 12(b) and Remark 4, we get

$$I_1 \leqslant C \int_0^{\frac{1}{4}|g^{-1}g_0|^2} \int_{|g^{-1}h|<\sqrt{s}} \left(\int_{B(g_0,r)} s^{-\frac{Q}{2}} e^{-As^{-1}|g^{-1}g_0|^2} \left(\frac{r}{\sqrt{s}}\right)^{\delta'} \left|a(w)\right| dw \right)^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}$$
$$\leqslant C \int_0^{\frac{1}{4}|g^{-1}g_0|^2} s^{-Q} \left(\frac{|g^{-1}g_0|}{\sqrt{s}}\right)^{-2(Q+1)} \left(\frac{r}{\sqrt{s}}\right)^{2\delta'} \frac{ds}{s} = \frac{Cr^{2\delta'}}{|g^{-1}g_0|^{2(Q+\delta')}}.$$

For any $w \in B(g_0, r)$ and $g \notin B(g_0, 4r)$, when $\sqrt{s} \geqslant \frac{1}{2}|g^{-1}g_0|$, we have $|w^{-1}g_0| \leqslant \sqrt{s}$. By Lemma 12(b) again, we get

$$I_2 \leqslant C \int_{\frac{1}{4}|g^{-1}g_0|^2}^{\infty} \int_{|g^{-1}h|<\sqrt{s}} \left(\int_{B(g_0,r)} s^{-\frac{Q}{2}} \left(\frac{r}{\sqrt{s}}\right)^{\delta'} \left|a(w)\right| dw \right)^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}$$
$$\leqslant C \int_{\frac{1}{4}|g^{-1}g_0|^2}^{\infty} s^{-Q} \left(\frac{r}{\sqrt{s}}\right)^{2\delta'} \frac{ds}{s} = \frac{Cr^{2\delta'}}{|g^{-1}g_0|^{2(Q+\delta')}}.$$

Thus,

$$\int_{|g^{-1}g_0|\geqslant 4r} S_Q^L a(g)\,dg \leqslant C \int_{|g^{-1}g_0|\geqslant 4r} \frac{r^{\delta'}}{|g^{-1}g_0|^{(Q+\delta')}}\,dg = C. \tag{27}$$

Now we consider the case $r \geqslant \rho(g_0)$.

$$S_Q^L a(g)^2 = \int_0^{\frac{1}{4}|g^{-1}g_0|^2} \int_{|g^{-1}h|<\sqrt{s}} \left| \int_{B(g_0,r)} Q_s^L(h,w)a(w)\,dw \right|^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}$$
$$+ \int_{\frac{1}{4}|g^{-1}g_0|^2}^{\infty} \int_{|g^{-1}h|<\sqrt{s}} \left| \int_{B(g_0,r)} Q_s^L(h,w)a(w)\,dw \right|^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}$$
$$= J_1 + J_2.$$

Note that $\rho(w) \leqslant Cr$ for any $w \in B(g_0, r)$. Similar to the estimate of I_1, using Lemma 12(a), we get

$$J_1 \leqslant C \int_0^{\frac{1}{4}|g^{-1}g_0|^2} \int_{|g^{-1}h|<\sqrt{s}} \left(\int_{B(g_0,r)} s^{-\frac{Q}{2}} e^{-As^{-1}|g^{-1}g_0|^2} \frac{\rho(w)}{\sqrt{s}} \left|a(w)\right| dw \right)^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}$$
$$\leqslant C \int_0^{\frac{1}{4}|g^{-1}g_0|^2} s^{-Q} \left(\frac{|g^{-1}g_0|}{\sqrt{s}}\right)^{-2(Q+2)} \left(\frac{r}{\sqrt{s}}\right)^{2} \frac{ds}{s} = \frac{Cr^{2}}{|g^{-1}g_0|^{2(Q+1)}}$$

and

$$J_2 \leqslant C \int_{\frac{1}{4}|g^{-1}g_0|^2}^{\infty} \int_{|g^{-1}h|<\sqrt{s}} \left(\int_{B(g_0,r)} s^{-\frac{Q}{2}} \frac{\rho(w)}{\sqrt{s}} \left|a(w)\right| dw \right)^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}$$

$$\leqslant C \int_{\frac{1}{4}|g^{-1}g_0|^2}^{\infty} s^{-Q} \left(\frac{r}{\sqrt{s}} \right)^2 \frac{ds}{s} = \frac{Cr^2}{|g^{-1}g_0|^{2(Q+1)}}.$$

Therefore,

$$\int_{|g^{-1}g_0| \geqslant 4r} S_Q^L a(g)\,dg \leqslant C \int_{|g^{-1}g_0| \geqslant 4r} \frac{r}{|g^{-1}g_0|^{(Q+1)}}\,dg = C. \tag{28}$$

The estimate for $\|S_Q^L a\|_{L^1}$ follows from the combination of (26)–(28). □

Lemma 15. *The operators s_Q^L and S_Q^L are bounded from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$.*

Proof. As same as Lemma 14, we only give the proof for S_Q^L. By the Calderón–Zygmund decomposition (cf. [17, Chapter 1, §4]), given $f \in L^1(\mathbb{H}^n)$ and $\alpha > 0$, we have the decomposition $f = f_1 + f_2$, with $f_2 = \sum_j b_j$, such that

(i) $|f_1(g)| \leqslant C\alpha$ for almost everywhere $g \in \mathbb{H}^n$;
(ii) each b_j is supported on a ball B_j,

$$\int_{B_j} \left|b_j(g)\right| dg \leqslant C\alpha |B_j| \quad \text{and} \quad \int_{B_j} b_j(g)\,dg = 0;$$

(iii) $\{B_j\}$ has finite overlaps property and $\sum_j |B_j| \leqslant \frac{C}{\alpha}\|f\|_{L^1}$.

It is clear that

$$\left|\left\{ g \in \mathbb{H}^n \colon S_Q^L f_1(g) > \frac{\alpha}{2} \right\}\right| \leqslant \frac{C}{\alpha^2}\|f_1\|_{L^2}^2 \leqslant \frac{C}{\alpha}\|f\|_{L^1}. \tag{29}$$

Let $B_j = B(g_j, r_j)$ and $E = \bigcup_j B(g_j, 4r_j)$. Then

$$|E| \leqslant C \sum_j |B_j| \leqslant \frac{C}{\alpha}\|f\|_{L^1}. \tag{30}$$

By the same arguments as (27) and (28), we have

$$\int\limits_{|g^{-1}g_j|\geqslant 4r_j} S_Q^L b_j(g)\,dg \leqslant C \int\limits_{B_j} \left|b_j(g)\right| dg \leqslant C\alpha |B_j|$$

which implies

$$\begin{aligned}\left|\left\{g \notin E\colon\ S_Q^L f_2(g) > \frac{\alpha}{2}\right\}\right| &\leqslant \frac{C}{\alpha} \int\limits_{E^c} S_Q^L f_2(g)\,dg \\ &\leqslant \frac{C}{\alpha} \sum_j \int\limits_{|g^{-1}g_j|\geqslant 4r_j} S_Q^L b_j(g)\,dg \\ &\leqslant \frac{C}{\alpha}\|f\|_{L^1}. \end{aligned} \tag{31}$$

The combination of (29)–(31) gives

$$\left|\left\{g \in \mathbb{H}^n\colon\ S_Q^L f(g) > \alpha\right\}\right| \leqslant \frac{C}{\alpha}\|f\|_{L^1}.$$

This proves that S_Q^L is bounded from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$. $\square$

As showed in [2], in general, it is not enough to conclude that an operator extends to a bounded operator on the whole Hardy space by verifying that it is bounded on atoms. To establish the (H_L^1, L^1) boundedness, we use the following lemma (cf. [14, Lemma 18]).

Lemma 16. *If T is a bounded sublinear operator from $L^1(\mathbb{H}^n)$ to $L^{1,\infty}(\mathbb{H}^n)$ and satisfies $\|Ta\|_{L^1} \leqslant C$ for any $H_L^{1,\infty}$-atom a, then T is bounded from $H_L^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$.*

From Lemmas 14–16, we obtain

Lemma 17. *The operators s_Q^L and S_Q^L are bounded from $H_L^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$.*

Now Theorem 4 follows from the combination of the above lemmas. In fact, Theorem 6 which will be proved in Section 8 implies that s_Q^L and S_Q^L are bounded from $L^\infty(\mathbb{H}^n)$ to $BMO(\mathbb{H}^n)$. By an interpolation argument, s_Q^L and S_Q^L are bounded on $L^p(\mathbb{H}^n)$ for $1 < p < \infty$ as pointed out in Remark 3. In view of Lemma 13, s_Q^L and S_Q^L are isometric operators on $L^2(\mathbb{H}^n)$ up to constant factors. The reverse estimates are obtained by duality.

7. The Carleson measure characterization

In this section we prove the Carleson measure characterization of $BMO_L(\mathbb{H}^n)$. First we prove Theorem 5 which extends the duality inequality of tent spaces to the Siegel upper half-space $\mathbf{U}^n$.

Proof of Theorem 5. The idea of the proof is the same as in [5]. Recall the definition of the Carleson box $\Omega(B) = \Omega(g, r)$ based on B defined in Section 2

$$\Omega(g,r)=\{(h,s)\in\mathbf{U}^n\colon\ |g^{-1}h|<r,\ 0<s<r^2\}.$$

For any $\tau>0$, let

$$\Gamma^\tau(g)=\{(h,s)\in\mathbf{U}^n\colon\ |g^{-1}h|<\sqrt{s},\ 0<s<\tau^2\}$$

and set

$$\mathcal{A}(F|\tau)(g)=\left(\int\limits_{\Gamma^\tau(g)}|F(h,s)|^2\frac{dh\,ds}{s^{\frac{Q}{2}+1}}\right)^{\frac{1}{2}}.$$

Obviously, $\mathcal{A}(F|\tau)$ is increasing with τ and $\mathcal{A}(F|\infty)=\mathcal{A}(F)$. Let B be any ball of radius r and B^* the concentric ball with radius $2r$. We have $\bigcup_{g\in B}\Gamma^r(g)\subset\Omega(B^*)$. Thus,

$$\int\limits_B\mathcal{A}(\Phi|r)(g)^2\,dg\leqslant c_n\int\limits_{\Omega(B^*)}|\Phi(h,s)|^2\frac{dh\,ds}{s},$$

and hence

$$\frac{1}{|B|}\int\limits_B\mathcal{A}(\Phi|r)(g)^2\,dg\leqslant\frac{2^Qc_n}{|B^*|}\int\limits_{\Omega(B^*)}|\Phi(h,s)|^2\frac{dh\,ds}{s}\leqslant 2^Qc_n\inf_{g\in B}\mathcal{C}(\Phi)(g)^2,\tag{32}$$

where c_n is the volume of the unit ball. Write $A_1=2^{\frac{Q+1}{2}}\sqrt{c_n}$. For given Φ, we define the "stopping time" $\tau(g)$ by

$$\tau(g)=\sup\{\tau>0\colon\ \mathcal{A}(\Phi|\tau)(g)\leqslant A_1\mathcal{C}(\Phi)(g)\}.$$

By (32), we then have

$$\left|\{g\in B\colon\ \tau(g)\geqslant r\}\right|\geqslant\frac{1}{2}|B|.$$

Therefore, by Fubini's Theorem and Schwarz's inequality, we obtain

$$\begin{aligned}\int\limits_{\mathbf{U}^n}|F(g,s)\Phi(g,s)|\frac{dg\,ds}{s}&\leqslant C\int\limits_{\mathbb{H}^n}\left(\int\limits_{\Gamma^{\tau(g)}(g)}|F(h,s)\Phi(h,s)|\frac{dh\,ds}{s^{\frac{Q}{2}+1}}\right)dg\\&\leqslant C\int\limits_{\mathbb{H}^n}\mathcal{A}\big(F|\tau(g)\big)(g)\mathcal{A}\big(\Phi|\tau(g)\big)(g)\,dg\\&\leqslant C\int\limits_{\mathbb{H}^n}\mathcal{A}(F)(g)\mathcal{C}(\Phi)(g)\,dg\\&\leqslant C\|\mathcal{A}(F)\|_{L^1}\|\mathcal{C}(\Phi)\|_{L^\infty},\end{aligned}$$

which completes the proof of Theorem 5. $\square$

We are ready to prove Theorem 3.

Proof of Theorem 3. Let $\varphi \in BMO_L(\mathbb{H}^n)$. Then φ satisfies (1) and, by Lemma 12(a),

$$Q_s^L\varphi(g) = \int_{\mathbb{H}^n} Q_s^L(g,h)\varphi(h)\,dh$$

is absolutely convergent. To prove assertion (a), we need to prove that, for any ball $B = B(g_0, r)$,

$$\frac{1}{|B|}\int_{\Omega(B)} \left|Q_s^L\varphi(g)\right|^2 \frac{dg\,ds}{s} \leqslant C\|\varphi\|_{BMO_L}^2. \tag{33}$$

Set $B_k = B(g_0, 2^k r)$ and

$$\varphi = \big(\varphi - \varphi(B_1)\big)\chi_{B_1} + \big(\varphi - \varphi(B_1)\big)\chi_{(B_1)^c} + \varphi(B_1) := \widetilde{\varphi}_1 + \widetilde{\varphi}_2 + \varphi(B_1).$$

By Lemma 13,

$$\begin{aligned}
\frac{1}{|B|}\int_{\Omega(B)} \left|Q_s^L\widetilde{\varphi}_1(g)\right|^2 \frac{dg\,ds}{s} &\leqslant \frac{1}{|B|}\int_B \left|s_Q^L\widetilde{\varphi}_1(g)\right|^2 dg \\
&\leqslant \frac{1}{4|B|}\|\widetilde{\varphi}_1\|_{L^2}^2 = \frac{1}{4|B|}\int_{B_1}\left|\varphi(g) - \varphi(B_1)\right|^2 dg \\
&\leqslant C\|\varphi\|_{BMO}^2 \leqslant C\|\varphi\|_{BMO_L}^2.
\end{aligned} \tag{34}$$

Note that

$$\left|\varphi(B_{k+1}) - \varphi(B_1)\right| \leqslant Ck\|\varphi\|_{BMO}.$$

For $g \in B(g_0, r)$, by Lemma 12(a),

$$\begin{aligned}
\left|Q_s^L\widetilde{\varphi}_2(g)\right| &\leqslant C\int_{(B_1)^c} s^{-\frac{Q}{2}}\left(\frac{|h^{-1}g|}{\sqrt{s}}\right)^{-(Q+1)}\left|\widetilde{\varphi}_2(h)\right| dh \\
&\leqslant C\frac{\sqrt{s}}{r}\sum_{k=1}^{\infty} 2^{-k}\left(\frac{1}{|B_{k+1}|}\int_{B_{k+1}\setminus B_k}\left|\varphi - \varphi(B_{k+1})\right| dh + \left|\varphi(B_{k+1}) - \varphi(B_1)\right|\right) \\
&\leqslant C\frac{\sqrt{s}}{r}\sum_{k=1}^{\infty} 2^{-k}(1+k)\|\varphi\|_{BMO} \leqslant C\frac{\sqrt{s}}{r}\|\varphi\|_{BMO}.
\end{aligned}$$

Thus we have

$$\frac{1}{|B|} \int\limits_{\Omega(B)} \left| Q_s^L \widetilde{\varphi}_2(g) \right|^2 \frac{dg\, ds}{s} \leqslant C \|\varphi\|_{BMO}^2. \tag{35}$$

It remains to estimate the constant term. Assume first that $r < \rho(g_0)$. Choosing k_0 such that $2^{k_0} r < \rho(g_0) \leqslant 2^{k_0+1} r$, we have

$$\begin{aligned} \left|\varphi(B_1)\right| &\leqslant \left|\varphi(B_{k_0+1}) - \varphi(B_1)\right| + \left|\varphi(B_{k_0+1})\right| \\ &\leqslant C k_0 \|\varphi\|_{BMO} + \|\varphi\|_{BMO_L} \leqslant C\left(1 + \log \frac{\rho(g_0)}{r}\right) \|\varphi\|_{BMO_L}. \end{aligned}$$

Note that $\rho(g) \sim \rho(g_0) > r$ for any $g \in B(g_0, r)$. Using of Lemma 12(c), we get

$$\begin{aligned} \frac{1}{|B|} \int\limits_{\Omega(B)} \left| Q_s^L \big(\varphi(B_1)\mathbf{1}\big)(g) \right|^2 \frac{dg\, ds}{s} &= \frac{|\varphi(B_1)|^2}{|B|} \int\limits_{\Omega(B)} \left| \int\limits_{\mathbb{H}^n} Q_s^L(g,h)\, dh \right|^2 \frac{dg\, ds}{s} \\ &\leqslant \frac{C|\varphi(B_1)|^2}{|B|} \int\limits_{\Omega(B)} \left(\frac{\sqrt{s}}{\rho(g_0)} \right)^{2\delta} \frac{dg\, ds}{s} \\ &\leqslant C \|\varphi\|_{BMO_L}^2 \left(1 + \log \frac{\rho(g_0)}{r}\right)^2 \left(\frac{r}{\rho(g_0)} \right)^{2\delta} \\ &\leqslant C \|\varphi\|_{BMO_L}^2. \end{aligned} \tag{36}$$

For $r \geqslant \rho(g_0)$, we have $|\varphi(B_1)| \leqslant \|\varphi\|_{BMO_L}$. By Lemma 12(c) again,

$$\begin{aligned} &\frac{1}{|B|} \int\limits_{\Omega(B)} \left| Q_s^L \big(\varphi(B_1)\mathbf{1}\big)(g) \right|^2 \frac{dg\, ds}{s} \\ &\leqslant \frac{|\varphi(B_1)|^2}{|B|} \int\limits_B \int\limits_0^\infty \left| \int\limits_{\mathbb{H}^n} Q_s^L(g,h)\, dh \right|^2 \frac{ds}{s}\, dg \\ &\leqslant \frac{C|\varphi(B_1)|^2}{|B|} \left(\int\limits_B \int\limits_0^{\rho(g)^2} \left(\frac{\sqrt{s}}{\rho(g)} \right)^{2\delta} \frac{ds}{s}\, dg + \int\limits_B \int\limits_{\rho(g)^2}^\infty \left(\frac{\sqrt{s}}{\rho(g)} \right)^{-2} \frac{ds}{s}\, dg \right) \\ &\leqslant C \|\varphi\|_{BMO_L}^2. \end{aligned} \tag{37}$$

Then (33) follows from (34)–(37), and we prove part (a).

Before proving assertion (b), we claim that for $\varphi \in L^2(\mathbb{H}^n)$,

$$\varphi = 4 \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int\limits_\varepsilon^N \big(Q_s^L\big)^2 \varphi \, \frac{ds}{s} \quad \text{in } L^2\big(\mathbb{H}^n\big). \tag{38}$$

By the spectral theorem, we write Q_s^L in the form

$$Q_s^L = -\int_0^\infty s\lambda e^{-s\lambda}\, dE(\lambda),$$

where $\{E(\lambda)\}$ is a resolution of the identity. By functional calculus, for $0 < N_1 < N_2 < \infty$,

$$\left\| \int_{N_1}^{N_2} \left(Q_s^L\right)^2 \varphi \,\frac{ds}{s} \right\|_{L^2(\mathbb{H}^n)}^2 = \int_0^\infty \left(\int_{N_1}^{N_2} s^2 \lambda^2 e^{-2s\lambda} \,\frac{ds}{s} \right)^2 \langle dE(\lambda)\varphi, \varphi \rangle.$$

Note that 0 is not an eigenvalue of L because $V(g) > 0$ for almost every g. It is easy to see that, for any $\lambda > 0$,

$$\lim_{N_1, N_2 \to \infty} \int_{N_1}^{N_2} s^2 \lambda^2 e^{-2s\lambda} \,\frac{ds}{s} = 0.$$

The dominated convergence theorem gives

$$\lim_{N_1, N_2 \to \infty} \left\| \int_{N_1}^{N_2} \left(Q_s^L\right)^2 \varphi \,\frac{ds}{s} \right\|_{L^2(\mathbb{H}^n)}^2 = 0.$$

Similarly, for $0 < \varepsilon_1 < \varepsilon_2 < \infty$,

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \left\| \int_{\varepsilon_1}^{\varepsilon_2} \left(Q_s^L\right)^2 \varphi \,\frac{ds}{s} \right\|_{L^2(\mathbb{H}^n)}^2 = 0.$$

Therefore,

$$\widetilde{\varphi} := 4 \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \left(Q_s^L\right)^2 \varphi \,\frac{ds}{s} \quad \text{in } L^2\left(\mathbb{H}^n\right)$$

is well defined. Since Q_s^L is self-adjoint, using the polarized version of Lemma 13, we get

$$\begin{aligned}
\langle \widetilde{\varphi}, \psi \rangle &= 4 \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \left\langle Q_s^L \varphi, Q_s^L \psi \right\rangle \frac{ds}{s} \\
&= 4 \int_0^\infty \left\langle Q_s^L \varphi, Q_s^L \psi \right\rangle \frac{ds}{s} \\
&= \langle \varphi, \psi \rangle
\end{aligned}$$

for any $\psi \in L^2(\mathbb{H}^n)$. Thus $\widetilde{\varphi} = \varphi$ and (38) is established.

Now we prove assertion (b). Suppose that $f \in L_c^\infty$, φ satisfies (1), and $d\mu_\varphi$ is a Carleson measure. Set

$$F(g,s) = Q_s^L f(g) \quad \text{and} \quad \Phi(g,s) = Q_s^L \varphi(g), \quad (g,s) \in \mathbf{U}^n.$$

By Lemma 17,

$$\|\mathcal{A}(F)\|_{L^1} = \|S_Q^L f\|_{L^1} \leqslant C\|f\|_{H_L^1}.$$

Also note that

$$\|\mathcal{C}(\Phi)\|_{L^\infty} = \|d\mu_\varphi\|_{\mathcal{C}}^{\frac{1}{2}}.$$

We claim that

$$\int_{\mathbb{H}^n} f(g)\overline{\varphi(g)}\,dg = 4\int_{\mathbf{U}^n} F(g,s)\overline{\Phi(g,s)}\,\frac{dg\,ds}{s}. \tag{39}$$

Then assertion (b) is easily derived from (39), Theorems 5 and 1. Roughly speaking, we have

$$\begin{aligned}
4\int_{\mathbf{U}^n} F(g,s)\overline{\Phi(g,s)}\,\frac{dg\,ds}{s} &= 4\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \int_{\mathbb{H}^n} Q_s^L f(g)\overline{Q_s^L\varphi(g)}\,\frac{dg\,ds}{s} \\
&= 4\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} Q_s^L f(g) Q_s^L(h,g)\overline{\varphi(h)}\,\frac{dg\,dh\,ds}{s} \\
&= 4\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \int_{\mathbb{H}^n} \left(Q_s^L\right)^2 f(h)\overline{\varphi(h)}\,\frac{dh\,ds}{s} \\
&= 4\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbb{H}^n} \int_\varepsilon^N \left(Q_s^L\right)^2 f(h)\overline{\varphi(h)}\,\frac{ds\,dh}{s} \\
&= \int_{\mathbb{H}^n} f(h)\overline{\varphi(h)}\,dh.
\end{aligned} \tag{40}$$

Here we may use subsequences $\varepsilon \to 0$ and $N_k \to \infty$ to replace $\varepsilon \to 0$ and $N \to \infty$, respectively, if necessary. In order to justify these steps, we have to verify the absolute convergence of the above integrals. Suppose f is supported in $B(0,r)$. By Lemma 12(a), if $g \in B(0,2r)$,

$$\left|Q_s^L f(g)\right| \leqslant C\|f\|_{L^\infty}.$$

If $g \notin B(0,2r)$, by Lemma 4, $\rho(h) \leqslant C\max\{\rho(0), r\}$ for any $h \in B(0,r)$, then

$$\left|Q_s^L f(g)\right| \leqslant C \frac{\max\{\rho(0), r\}}{\sqrt{s}} s^{-\frac{Q}{2}} e^{-As^{-1}|g|^2} \|f\|_{L^1}.$$

Hence

$$\sup_{s>0}\left|Q_s^L f(g)\right| \leqslant C_f\left(1+|g|\right)^{-(Q+1)} \quad \text{for all } g \in \mathbb{H}^n, \tag{41}$$

which implies

$$\int_{\mathbb{H}^n}\left|Q_s^L(h,g) Q_s^L f(g)\right| dg \leqslant C_{f,s}\left(1+|h|\right)^{-(Q+1)}.$$

Let $(Q_s^L)^2(g,h)$ denote the kernel of $(Q_s^L)^2$. By Lemma 12(a), we have

$$\begin{aligned}
\left|\left(Q_s^L\right)^2(g,h)\right| &= \left|\int_{\mathbb{H}^n} Q_s^L(g,w) Q_s^L(w,h)\, dw\right| \\
&\leqslant C_N \int_{\mathbb{H}^n} s^{-Q} e^{-As^{-1}|w^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}\right)^{-N} e^{-As^{-1}|h^{-1}w|^2}\left(1+\frac{\sqrt{s}}{\rho(h)}\right)^{-N} dw \\
&\leqslant C_N s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}+\frac{\sqrt{s}}{\rho(h)}\right)^{-N}.
\end{aligned}$$

The same arguments as (41), we have

$$\sup_{s>0}\left|T_s^L f(g)\right| \leqslant C_f\left(1+|g|\right)^{-(Q+1)} \quad \text{for all } g \in \mathbb{H}^n;$$

$$\sup_{s>0}\left|\left(Q_s^L\right)^2 f(g)\right| \leqslant C_f\left(1+|g|\right)^{-(Q+1)} \quad \text{for all } g \in \mathbb{H}^n.$$

Let $P_\varepsilon(g,h)$ denote the kernel of $P_\varepsilon = \int_\varepsilon^\infty (Q_s^L)^2 \frac{ds}{s}$. By functional calculus,

$$\begin{aligned}
P_\varepsilon &= \int_0^\infty\left(\int_\varepsilon^\infty s^2\lambda^2 e^{-2s\lambda}\frac{ds}{s}\right) dE(\lambda) \\
&= \int_0^\infty\left(\frac{1}{2}\varepsilon\lambda e^{-2\varepsilon\lambda}+\frac{1}{4}e^{-2\varepsilon\lambda}\right) dE(\lambda) \\
&= -\frac{1}{4}Q_{2\varepsilon}^L+\frac{1}{4}T_{2\varepsilon}^L.
\end{aligned}$$

Thus $P_\varepsilon(g,h) = -\frac{1}{4}Q_{2\varepsilon}^L(g,h)+\frac{1}{4}T_{2\varepsilon}^L(g,h)$. Also we have

$$\sup_{0<\varepsilon<N<\infty}\left|\int_{\varepsilon}^{N}\left(Q_s^L\right)^2 f(g)\,\frac{ds}{s}\right| = \sup_{0<\varepsilon<N<\infty}\left|P_\varepsilon f(g) - P_N f(g)\right| \leqslant C_f\left(1+|g|\right)^{-(Q+1)}.$$

Hence all integrals in (40) are absolutely convergent. This proves assertion (b) and the proof of Theorem 3 is completed. □

8. BMO_L-boundedness

In this section we give the proof of Theorem 6.

First we consider the Hardy–Littlewood maximal function. Let $\varphi \in BMO_L(\mathbb{H}^n)$ and $M\varphi$ be the Hardy–Littlewood maximal function of φ. Given $B = B(g_0, r)$, as the beginning of Section 5, we set

$$\varphi = \varphi\chi_{B^*} + \varphi\chi_{(B^*)^c} = \varphi_1 + \varphi_2,$$

where $B^* = B(g_0, 2r)$. Suppose $r \geqslant \rho(g_0)$. Because the Hardy–Littlewood maximal function is bounded on $L^2(\mathbb{H}^n)$, similar to (21), we have

$$\frac{1}{|B|}\int_B M\varphi_1(g)\,dg \leqslant C\|\varphi\|_{BMO_L}.$$

For $g \in B(g_0, r)$, since $\rho(g) \leqslant Cr$,

$$\begin{aligned} M\varphi_2(g) &= \sup_{\substack{g\in B' \\ B'\cap(B^*)^c\neq\emptyset}} \frac{1}{|B'|}\int_{B'\cap(B^*)^c}\left|\varphi(h)\right|dh \\ &\leqslant \sup_{s\geqslant\rho(g)}\frac{C}{|B(g,s)|}\int_{B(g,s)}\left|\varphi(h)\right|dh \\ &\leqslant C\|\varphi\|_{BMO_L}. \end{aligned}$$

We then have

$$\frac{1}{|B|}\int_B M\varphi_2(g)\,dg \leqslant C\|\varphi\|_{BMO_L}.$$

Therefore,

$$\frac{1}{|B|}\int_B M\varphi(g)\,dg \leqslant C\|\varphi\|_{BMO_L} \quad \text{for } r \geqslant \rho(g_0). \tag{42}$$

The inequality (42) also shows that $\Phi = M\varphi$ is finite almost everywhere for $\varphi \in BMO_L(\mathbb{H}^n)$.

In case $r < \rho(g_0)$, it suffices to show

$$\frac{1}{|B|}\int_B \left|\Phi(g) - \Phi(B)\right| dg \leqslant C\|\varphi\|_{BMO}. \tag{43}$$

We may assume that φ is nonnegative. Let

$$\Phi_1(g) = \sup_{g\in B'\subset B^*} \frac{1}{|B'|}\int_{B'} \varphi(h)\,dh,$$
$$\Phi_2(g) = \sup_{\substack{g\in B' \\ B'\cap(B^*)^c\neq\emptyset}} \frac{1}{|B'|}\int_{B'} \varphi(h)\,dh.$$

Clearly $\Phi(g) = \max\{\Phi_1(g), \Phi_2(g)\}$. Set

$$E = \left\{g\in B\colon\ \Phi(g) > \Phi(B)\right\},$$
$$E_1 = \left\{g\in E\colon\ \Phi_1(g) \geqslant \Phi_2(g)\right\},$$
$$E_2 = \left\{g\in E\colon\ \Phi_1(g) < \Phi_2(g)\right\}.$$

We have

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left|\Phi(g) - \Phi(B)\right| dg \\
&\quad = \frac{2}{|B|}\int_E \left(\Phi(g) - \Phi(B)\right) dg \\
&\quad = \frac{2}{|B|}\int_{E_1} \left(\Phi_1(g) - \Phi(B)\right) dg + \frac{2}{|B|}\int_{E_2} \left(\Phi_2(g) - \Phi(B)\right) dg.
\end{aligned} \tag{44}$$

Set

$$\widetilde{\varphi}_1 = \left(\varphi - \varphi\left(B^*\right)\right)\chi_{B^*}.$$

Since $\varphi(B^*) \leqslant \Phi(B)$, we have $\Phi_1(g) \leqslant M\widetilde{\varphi}_1(g) + \Phi(B)$. Therefore,

$$\begin{aligned}
\frac{1}{|B|}\int_{E_1} \left(\Phi_1(g) - \Phi(B)\right) dg &\leqslant \frac{1}{|B|}\int_{E_1} M\widetilde{\varphi}_1(g)\,dg \\
&\leqslant \left(\frac{1}{|B|}\int_{E_1} M\widetilde{\varphi}_1(g)^2\,dg\right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{C}{|B^*|}\int_{B^*} \widetilde{\varphi}_1(g)^2\,dg\right)^{\frac{1}{2}} \\
&\leqslant C\|\varphi\|_{BMO}.
\end{aligned} \tag{45}$$

On the other hand, let $g \in E_2$ and B' be any ball satisfying $g \in B'$ and $B' \cap (B^*)^c \neq \emptyset$. Let $B'' = 8B'$; that is, the ball concentric with B' whose radius is 8 times of B''s. Then $B \subset B''$ and $\varphi(B'') \leqslant \Phi(B)$. Thus we have

$$\varphi(B') - \Phi(B) \leqslant \varphi(B') - \varphi(B'') \leqslant C\|\varphi\|_{BMO}.$$

Taking the supremum over all such balls B', we obtain

$$\Phi_2(g) - \Phi(B) \leqslant C\|\varphi\|_{BMO},$$

and hence

$$\frac{1}{|B|}\int_{E_2} \big(\Phi_2(g) - \Phi(B)\big)\,dg \leqslant C\|\varphi\|_{BMO}. \tag{46}$$

The combination of (44)–(46) gives (43). This proves

$$\|M\varphi\|_{BMO_L} \leqslant C\|\varphi\|_{BMO_L}.$$

Next we deal with the semigroup maximal function $T_L^*\varphi$. Given $B = B(g_0, r)$, if $r \geqslant \rho(g_0)$, the same argument as [18, p. 57, (16)] and (42) yield

$$\frac{1}{|B|}\int_B T_L^*\varphi(g)\,dg \leqslant \frac{C}{|B|}\int_B M\varphi(g)\,dg \leqslant C\|\varphi\|_{BMO_L}. \tag{47}$$

Suppose $r < \rho(g_0)$. We will prove that

$$\sup_{s>0}\big|T_s^L\varphi(g) - T_s\varphi(g)\big| \leqslant C\|\varphi\|_{BMO_L} \tag{48}$$

and there exists a constant A_1 depending on B such that

$$\frac{1}{|B|}\int_B \big|T^*\varphi(g) - A_1\big|\,dg \leqslant C\|\varphi\|_{BMO}, \tag{49}$$

where $T^*\varphi$ is the heat maximal function defined by

$$T^*\varphi(g) = \sup_{s>0}\big|T_s\varphi(g)\big|.$$

It is easy to deduce from (48) and (49) that

$$\frac{1}{|B|}\int_B \big|T_L^*\varphi(g) - A_1\big|\,dg \leqslant C\|\varphi\|_{BMO_L}. \tag{50}$$

Thus, (47) and (50) yield

$$\left\|T_L^*\varphi\right\|_{BMO_L} \leqslant C\|\varphi\|_{BMO_L}.$$

To prove (48), set $B_k = B(g, 2^k\sqrt{s})$, $k = 1, 2, \ldots$. If $s \geqslant \rho(g)^2$, then by (4),

$$\begin{aligned}
\left|T_s^L\varphi(g) - T_s\varphi(g)\right| &\leqslant \int_{\mathbb{H}^n} \left(K_s^L(g,h) + H_s\left(h^{-1}g\right)\right)\left|\varphi(h)\right| dh \\
&\leqslant C\sum_{k=1}^{\infty} s^{-\frac{Q}{2}} 2^{-k(Q+1)} \int_{B_k} \left|\varphi(h)\right| dh \\
&\leqslant C\sum_{k=1}^{\infty} 2^{-k}\|\varphi\|_{BMO_L} \leqslant C\|\varphi\|_{BMO_L}.
\end{aligned}$$

If $s < \rho(g)^2$, for $1 \leqslant k \leqslant k_0$ where k_0 satisfies $2^{k_0}\sqrt{s} < \rho(g) \leqslant 2^{k_0+1}\sqrt{s}$, we have

$$\begin{aligned}
\int_{B_k} \left|\varphi(g)\right| dg &\leqslant \int_{B_k} \left|\varphi(g) - \varphi(B_k)\right| dg + |B_k|\left(\left|\varphi(B_k) - \varphi(B_{k_0+1})\right| + \left|\varphi(B_{k_0+1})\right|\right) \\
&\leqslant C(k_0+1)|B_k|\|\varphi\|_{BMO_L}.
\end{aligned}$$

By Lemma 9,

$$\begin{aligned}
\left|T_s^L\varphi(g) - T_s\varphi(g)\right| &\leqslant \int_{\mathbb{H}^n} E_s(g,h)\left|\varphi(h)\right| dh \\
&\leqslant C\sum_{k=1}^{k_0} s^{-\frac{Q}{2}} 2^{-k(Q+1)} 2^{-k_0\delta} \int_{B_k} \left|\varphi(h)\right| dh \\
&\quad + C\sum_{k=k_0+1}^{\infty} s^{-\frac{Q}{2}} 2^{-k(Q+1)} \int_{B_k} \left|\varphi(h)\right| dh \\
&\leqslant C(k_0+1)2^{-k_0\delta}\sum_{k=1}^{k_0} 2^{-k}\|\varphi\|_{BMO_L} + C\sum_{k=k_0+1}^{\infty} 2^{-k}\|\varphi\|_{BMO_L} \\
&\leqslant C\|\varphi\|_{BMO_L},
\end{aligned}$$

which proves (48).

To prove (49), we set

$$\varphi = \left(\varphi - \varphi\left(B^*\right)\right)\chi_{B^*} + \left(\varphi - \varphi\left(B^*\right)\right)\chi_{(B^*)^c} + \varphi\left(B^*\right) = \widetilde{\varphi}_1 + \widetilde{\varphi}_2 + \varphi\left(B^*\right),$$

where $B^* = B(g_0, 2r)$. We may assume that $T^*\widetilde{\varphi}_2(g_0) < \infty$ because $T^*\widetilde{\varphi}_2$ is finite almost everywhere. Set $\widetilde{\varphi}_2' = \widetilde{\varphi}_2 + \varphi(B^*)$. Since $T_s\mathbf{1} = \mathbf{1}$, $T^*\widetilde{\varphi}_2'(g_0)$ is finite. Hence

$$\begin{aligned}
&\frac{1}{|B|}\int\limits_B \left|T^*\varphi(g)-T^*\widetilde{\varphi}_2'(g_0)\right|dg\\
&\quad\leqslant \frac{1}{|B|}\int\limits_B T^*\widetilde{\varphi}_1(g)\,dg+\frac{1}{|B|}\int\limits_B \sup_{s>0}\left|T_s\widetilde{\varphi}_2'(g)-T_s\widetilde{\varphi}_2'(g_0)\right|dg\\
&\quad= \frac{1}{|B|}\int\limits_B T^*\widetilde{\varphi}_1(g)\,dg+\frac{1}{|B|}\int\limits_B \sup_{s>0}\left|T_s\widetilde{\varphi}_2(g)-T_s\widetilde{\varphi}_2(g_0)\right|dg.
\end{aligned}$$

Since T^* is bounded on $L^2(\mathbb{H}^n)$ (cf. [9]),

$$\begin{aligned}
\frac{1}{|B|}\int\limits_B T^*\widetilde{\varphi}_1(g)\,dg &\leqslant \left(\frac{1}{|B|}\int\limits_B \left(T^*\widetilde{\varphi}_1(g)\right)^2dg\right)^{\frac{1}{2}}\\
&\leqslant \left(\frac{C}{|B^*|}\int\limits_{B^*}\left|\widetilde{\varphi}_1(g)\right|^2dg\right)^{\frac{1}{2}}\\
&\leqslant C\|\varphi\|_{BMO}.
\end{aligned}$$

Let $\widetilde{B}_k=B(g_0,2^kr)$, $k=1,2,\ldots.$ Since

$$\left|\varphi(\widetilde{B}_{k+1})-\varphi(\widetilde{B}_1)\right|\leqslant Ck\|\varphi\|_{BMO},$$

we have

$$\begin{aligned}
\int\limits_{\widetilde{B}_{k+1}}\left|\widetilde{\varphi}_2(h)\right|dh &\leqslant \int\limits_{\widetilde{B}_{k+1}}\left|\varphi(h)-\varphi(\widetilde{B}_{k+1})\right|dh+|\widetilde{B}_{k+1}|\left|\varphi(\widetilde{B}_{k+1})-\varphi(\widetilde{B}_1)\right|\\
&\leqslant C(k+1)|\widetilde{B}_{k+1}|\|\varphi\|_{BMO}.
\end{aligned}$$

For $g\in B$, by (15), we get

$$\begin{aligned}
\left|T_s\widetilde{\varphi}_2(g)-T_s\widetilde{\varphi}_2(g_0)\right| &\leqslant \int\limits_{(B^*)^c}\left|H_s\left(h^{-1}g\right)-H_s\left(h^{-1}g_0\right)\right|\left|\widetilde{\varphi}_2(h)\right|dh\\
&\leqslant C\sum_{k=1}^{\infty}r^{-Q}2^{-k(Q+1)}\int\limits_{\widetilde{B}_{k+1}}\left|\widetilde{\varphi}_2(h)\right|dh\\
&\leqslant C\sum_{k=1}^{\infty}(k+1)2^{-k}\|\varphi\|_{BMO}\leqslant C\|\varphi\|_{BMO}.
\end{aligned}$$

Therefore,

$$\frac{1}{|B|}\int\limits_B\sup_{s>0}\left|T_s\widetilde{\varphi}_2(g)-T_s\widetilde{\varphi}_2(g_0)\right|dg\leqslant C\|\varphi\|_{BMO},$$

and (49) is proved.

Finally we establish the BMO_L-boundedness of the operators s_Q^L and S_Q^L. Let $\varphi \in BMO_L(\mathbb{H}^n)$ and $B = B(g_0, r)$. If $r \geqslant \rho(g_0)$, as the beginning of Section 5, we set

$$\varphi = \varphi\chi_{B^*} + \varphi\chi_{(B^*)^c} = \varphi_1 + \varphi_2,$$

where $B^* = B(g_0, 2r)$. In view of Lemma 13, similar to (21), we have

$$\frac{1}{|B|}\int_B S_Q^L\varphi_1(g)\,dg \leqslant C\left(\frac{1}{|B^*|}\int_{B^*}\left|\varphi(g)\right|^2 dg\right)^{\frac{1}{2}} \leqslant C\|\varphi\|_{BMO_L}. \tag{51}$$

By Lemma 12, if $|g^{-1}h| < \sqrt{s}$,

$$\begin{aligned}\left|Q_s^L(h,w)\right| &\leqslant \left|Q_s^L(h,w) - Q_s^L(g,w)\right| + \left|Q_s^L(g,w)\right| \\ &\leqslant Cs^{-\frac{Q}{2}}e^{-As^{-1}|w^{-1}g|^2}\left(1+\frac{\sqrt{s}}{\rho(g)}\right)^{-2}.\end{aligned} \tag{52}$$

We also note that $\rho(g) \leqslant Cr$ for any $g \in B(g_0, r)$. Then we get

$$\begin{aligned}S_Q^L\varphi_2(g)^2 &\leqslant \int_0^\infty \int_{|g^{-1}h|<\sqrt{s}} \left(\int_{(B^*)^c}\left|Q_s^L(h,w)\right|\left|\varphi(w)\right|dw\right)^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}} \\
&\leqslant C\int_0^\infty\left(\int_{(B^*)^c} s^{-\frac{Q}{2}}e^{-As^{-1}|w^{-1}g|^2}\left(1+\frac{\sqrt{s}}{r}\right)^{-2}\left|\varphi(w)\right|dw\right)^2\frac{ds}{s} \\
&\leqslant C\int_0^{r^2}\left(\int_{(B^*)^c} s^{-\frac{Q}{2}}\left(\frac{|w^{-1}g_0|}{\sqrt{s}}\right)^{-(Q+1)}\left|\varphi(w)\right|dw\right)^2\frac{ds}{s} \\
&\quad + C\int_{r^2}^\infty\left(\int_{(B^*)^c} s^{-\frac{Q}{2}}\left(\frac{|w^{-1}g_0|}{\sqrt{s}}\right)^{-(Q+1)}\left(\frac{\sqrt{s}}{r}\right)^{-2}\left|\varphi(w)\right|dw\right)^2\frac{ds}{s} \\
&\leqslant Cr^2\left(\int_{(B^*)^c}\left|w^{-1}g_0\right|^{-(Q+1)}\left|\varphi(w)\right|dw\right)^2 \\
&\leqslant C\left(\sum_{k=1}^\infty r^{-Q}2^{-k(Q+1)}\int_{2^k r\leqslant|w^{-1}g_0|<2^{k+1}r}\left|\varphi(w)\right|dw\right)^2 \\
&\leqslant C\left(\sum_{k=1}^\infty 2^{-k}\|\varphi\|_{BMO_L}\right)^2 = C\|\varphi\|_{BMO_L}^2.\end{aligned}$$

Hence

$$\frac{1}{|B|}\int_B S_Q^L\varphi_2(g)\,dg \leqslant C\|\varphi\|_{BMO_L}. \tag{53}$$

It follows from (51) and (53) that

$$\frac{1}{|B|}\int_B S_Q^L\varphi(g)\,dg \leqslant C\|\varphi\|_{BMO_L}.$$

A similar argument gives

$$\frac{1}{|B|}\int_B s_Q^L\varphi(g)\,dg \leqslant C\|\varphi\|_{BMO_L}.$$

Now we deal with the case $r < \rho(g_0)$. We give the estimate for s_Q^L first. As the proof of (49), we set

$$\varphi = \big(\varphi - \varphi(B^*)\big)\chi_{B^*} + \big(\varphi - \varphi(B^*)\big)\chi_{(B^*)^c} + \varphi(B^*) = \widetilde{\varphi}_1 + \widetilde{\varphi}_2 + \varphi(B^*),$$

where $B^* = B(g_0, 2r)$. Let s_Q denote the Littlewood–Paley function related to $\Delta_{\mathbb{H}^n}$ defined by

$$s_Q\varphi(g) = \left(\int_0^\infty \big|Q_s\varphi(g)\big|^2\,\frac{ds}{s}\right)^{\frac{1}{2}},$$

where $Q_s = s\frac{d}{ds}T_s$ has the convolution kernel $Q_s(g) = s\partial_s H_s(g)$. Set

$$A_2 = \left(\int_0^{\rho(g_0)^2} \big|Q_s\widetilde{\varphi}_2(g_0)\big|^2\,\frac{ds}{s}\right)^{\frac{1}{2}}.$$

It is easy to see that A_2 is a finite constant and

$$\begin{aligned}
\big|s_Q^L\varphi(g) - A_2\big| &\leqslant \left(\int_{\rho(g_0)^2}^\infty \big|Q_s^L\varphi(g)\big|^2\,\frac{ds}{s}\right)^{\frac{1}{2}} + \left|\left(\int_0^{\rho(g_0)^2} \big|Q_s\varphi(g)\big|^2\,\frac{ds}{s}\right)^{\frac{1}{2}} - A_2\right| \\
&\quad + \left(\int_0^{\rho(g_0)^2} \big|Q_s^L\varphi(g) - Q_s\varphi(g)\big|^2\,\frac{ds}{s}\right)^{\frac{1}{2}} \\
&= s_1(g) + s_2(g) + s_3(g).
\end{aligned}$$

Note that $\rho(g) \sim \rho(g_0)$ for any $g \in B(g_0, r)$. By Lemma 12(a), we get

$$\begin{aligned}
s_1(g)^2 &\leqslant \int_{\rho(g_0)^2}^{\infty} \left(\int_{\mathbb{H}^n} \left| Q_s^L(g,h) \right| \left| \varphi(h) \right| dh \right)^2 \frac{ds}{s} \\
&\leqslant C \int_{\rho(g_0)^2}^{\infty} \left(\int_{\mathbb{H}^n} s^{-\frac{Q}{2}} e^{-As^{-1}|h^{-1}g|^2} \left(1 + \frac{\sqrt{s}}{\rho(g)} \right)^{-2} \left| \varphi(h) \right| dh \right)^2 \frac{ds}{s} \\
&\leqslant C \int_{\rho(g_0)^2}^{\infty} \left(\int_{B(g,\rho(g))} \left| \varphi(h) \right| dh \right)^2 \frac{ds}{s^{Q+1}} \\
&\quad + C \int_{\rho(g_0)^2}^{\infty} \left(\int_{B(g,\rho(g))^c} \rho(g)^2 \left| h^{-1} g \right|^{-(Q+1)} \left| \varphi(h) \right| dh \right)^2 \frac{ds}{s^2} \\
&\leqslant C \left(\sum_{k=0}^{\infty} \rho(g)^{-Q} 2^{-k(Q+1)} \int_{B(g,2^k\rho(g))} \left| \varphi(h) \right| dh \right)^2 \\
&\leqslant C \|\varphi\|_{BMO_L}^2 \quad \text{for } g \in B(g_0, r).
\end{aligned} \tag{54}$$

Since $Q_s \mathbf{1} = 0$,

$$\begin{aligned}
s_2(g) &\leqslant \left(\int_0^{\rho(g_0)^2} \left| Q_s \widetilde{\varphi}_1(g) \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}} + \left(\int_0^{\rho(g_0)^2} \left| Q_s \widetilde{\varphi}_2(g) - Q_s \widetilde{\varphi}_2(g_0) \right|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&= s_{2,1}(g) + s_{2,2}(g).
\end{aligned}$$

It is known that s_Q is bounded on $L^2(\mathbb{H}^n)$ (cf. [9, Chapter 7]). Hence

$$\begin{aligned}
\frac{1}{|B|} \int_B s_{2,1}(g)\, dg &\leqslant \frac{1}{|B|} \int_B s_Q \widetilde{\varphi}_1(g)\, dg \\
&\leqslant \left(\frac{1}{|B|} \int_B s_Q \widetilde{\varphi}_1(g)^2\, dg \right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{C}{|B^*|} \int_{B^*} \left| \widetilde{\varphi}_1(g) \right|^2 dg \right)^{\frac{1}{2}} \\
&\leqslant C \|\varphi\|_{BMO}.
\end{aligned} \tag{55}$$

Note that the kernel $Q_s(g)$ satisfies the estimates

$$\left| Q_s(g) \right| \leqslant C s^{-\frac{Q}{2}} e^{-As^{-1}|g|^2}$$

and

$$\big|Q_s(gu)-Q_s(g)\big|\leqslant C|u|s^{-\frac{Q+1}{2}}e^{-As^{-1}|g|^2}\quad \text{if } |u|\leqslant\frac{|g|}{2}$$

(cf. [11, Theorem 1]). For $g\in B(g_0,r)$, we have

$$\begin{aligned}
s_{2,2}(g)^2 &\leqslant \int_0^{\rho(g_0)^2}\Bigg(\int_{(B^*)^c}\big|Q_s\big(h^{-1}g\big)-Q_s\big(h^{-1}g_0\big)\big|\big|\widetilde{\varphi}_2(h)\big|\,dh\Bigg)^2\frac{ds}{s}\\
&\leqslant C\int_0^{\rho(g_0)^2}\Bigg(\int_{(B^*)^c}\big|g_0^{-1}g\big|s^{-\frac{Q+1}{2}}e^{-As^{-1}|h^{-1}g_0|^2}\big|\widetilde{\varphi}_2(h)\big|\,dh\Bigg)^2\frac{ds}{s}\\
&\leqslant C\int_0^{r^2}\Bigg(\int_{(B^*)^c}r\big|h^{-1}g_0\big|^{-(Q+2)}\big|\widetilde{\varphi}_2(h)\big|\,dh\Bigg)^2 ds\\
&\quad +C\int_{r^2}^{\rho(g_0)^2}\Bigg(\int_{(B^*)^c}r\big|h^{-1}g_0\big|^{-(Q+\frac{1}{2})}\big|\widetilde{\varphi}_2(h)\big|\,dh\Bigg)^2\frac{ds}{s^{\frac{3}{2}}}\\
&\leqslant C\Bigg(\sum_{k=1}^{\infty}r^{-Q}2^{-k(Q+\frac{1}{2})}\int_{B(g_0,2^{k+1}r)}\big|\widetilde{\varphi}_2(h)\big|\,dh\Bigg)^2\\
&\leqslant C\Bigg(\sum_{k=1}^{\infty}(k+1)2^{-\frac{k}{2}}\|\varphi\|_{BMO_L}\Bigg)^2=C\|\varphi\|_{BMO_L}^2.
\end{aligned}\tag{56}$$

It remains to estimate $s_3(g)$. Let $G_s(g,h)=Q_s(h^{-1}g)-Q_s^L(g,h)$. We claim

$$\big|G_s(g,h)\big|\leqslant Cs^{-\frac{Q}{2}}e^{-As^{-1}|h^{-1}g|^2}\bigg(\frac{\sqrt{s}}{\rho(g)}\bigg)^{\delta}.\tag{57}$$

To show the claim, it suffices to consider the case $\sqrt{s}<\rho(g)$. It follows from the perturbation formula (12) that

$$\begin{aligned}
G_s(g,h)={}&\int_{\mathbb{H}^n}sH_{\frac{s}{2}}\big(w^{-1}g\big)V(w)K_{\frac{s}{2}}^L(w,h)\,dw\\
&+\int_0^{\frac{s}{2}}\int_{\mathbb{H}^n}\frac{s}{s-t}Q_{s-t}\big(w^{-1}g\big)V(w)K_t^L(w,h)\,dw\,dt
\end{aligned}$$

$$+\int_0^{\frac{s}{2}}\int_{\mathbb{H}^n} \frac{s}{s-t} H_t\big(w^{-1}g\big)V(w)Q_{s-t}^L(w,h)\,dw\,dt$$
$$= \widetilde{I}_0+\widetilde{I}_1+\widetilde{I}_2.$$

By the same argument as estimating I_1 in the proof of Lemma 9, we get

$$\widetilde{I}_1 \leqslant Cs^{-\frac{Q}{2}}e^{-As^{-1}|h^{-1}g|^2}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta},$$
$$\widetilde{I}_2 \leqslant Cs^{-\frac{Q}{2}}e^{-As^{-1}|h^{-1}g|^2}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta}.$$

Also, by Lemma 6, we have

$$\begin{aligned}
\widetilde{I}_0 &= \int_{|w^{-1}g|<\frac{|h^{-1}g|}{2}} sH_{\frac{s}{2}}\big(w^{-1}g\big)V(w)K_{\frac{s}{2}}^L(w,h)\,dw \\
&\quad+ \int_{|w^{-1}g|\geqslant\frac{|h^{-1}g|}{2}} sH_{\frac{s}{2}}\big(w^{-1}g\big)V(w)K_{\frac{s}{2}}^L(w,h)\,dw \\
&\leqslant Cs^{-\frac{Q}{2}+1}e^{-As^{-1}|h^{-1}g|^2}\int_{|w^{-1}g|<\frac{|h^{-1}g|}{2}} V(w)s^{-\frac{Q}{2}}e^{-As^{-1}|w^{-1}g|^2}\,dw \\
&\quad+ Cs^{-\frac{Q}{2}+1}\int_{|w^{-1}g|\geqslant\frac{|h^{-1}g|}{2}} V(w)s^{-\frac{Q}{2}}e^{-As^{-1}(|h^{-1}g|^2+|h^{-1}w|^2)}\,dw \\
&\leqslant Cs^{-\frac{Q}{2}}e^{-As^{-1}|h^{-1}g|^2}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta}.
\end{aligned}$$

Therefore the estimate (57) follows. For $g\in B(g_0,r)$, we have $\rho(g)\sim\rho(g_0)$ and get

$$\begin{aligned}
s_3(g)^2 &\leqslant \int_0^{\rho(g_0)^2}\left(\int_{\mathbb{H}^n}\big|G_s(g,h)\big|\big|\varphi(h)\big|\,dh\right)^2\frac{ds}{s} \\
&\leqslant C\int_0^{\rho(g_0)^2}\left(\int_{\mathbb{H}^n} s^{-\frac{Q}{2}}e^{-As^{-1}|h^{-1}g|^2}\left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta}\big|\varphi(h)\big|\,dh\right)^2\frac{ds}{s} \\
&\leqslant C\int_0^{\rho(g_0)^2}\left(\int_{B(g,\rho(g))}\rho(g)^{-\delta}\big|h^{-1}g\big|^{-(Q-\frac{\delta}{2})}\big|\varphi(h)\big|\,dh\right)^2\frac{ds}{s^{1-\frac{\delta}{2}}}
\end{aligned}$$

$$+ C\int_0^{\rho(g_0)^2}\left(\int_{B(g,\rho(g))^c} \left|h^{-1}g\right|^{-(Q+1)}\left|\varphi(h)\right| dh\right)^2 ds$$

$$\leqslant C\left(\sum_{k=0}^{\infty}\rho(g)^{-Q}2^{(k+1)(Q-\frac{\delta}{2})}\int_{B(g,2^{-k}\rho(g))}\left|\varphi(h)\right| dh\right)^2$$

$$+ C\left(\sum_{k=0}^{\infty}\rho(g)^{-Q}2^{-k(Q+1)}\int_{B(g,2^{k+1}\rho(g))}\left|\varphi(h)\right| dh\right)^2$$

$$\leqslant C\left(\sum_{k=0}^{\infty}(k+2)2^{-(k+1)\frac{\delta}{2}}+\sum_{k=0}^{\infty}2^{-k}\right)^2\|\varphi\|_{BMO_L}^2 \leqslant C\|\varphi\|_{BMO_L}^2. \tag{58}$$

By (54)–(56) and (58), we obtain

$$\frac{1}{|B|}\int_B \left|s_Q^L\varphi(g)-A_2\right| dg \leqslant C\|\varphi\|_{BMO_L}.$$

This gives the estimate for s_Q^L.

Then we give the estimate for S_Q^L. Set

$$\varphi = \left(\varphi-\varphi\left(B^{**}\right)\right)\chi_{B^{**}} + \left(\varphi-\varphi\left(B^{**}\right)\right)\chi_{(B^{**})^c} + \varphi\left(B^{**}\right) = \widetilde{\widetilde{\varphi}}_1 + \widetilde{\widetilde{\varphi}}_2 + \varphi\left(B^{**}\right),$$

where $B^{**} = B(g_0, 4r)$. Let S_Q denote the Lusin area integral related to $\Delta_{\mathbb{H}^n}$ defined by

$$S_Q\varphi(g) = \left(\int_{\Gamma(g)} \left|Q_s\varphi(h)\right|^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}\right)^{\frac{1}{2}}.$$

Set

$$A_3 = \left(\int_{\Gamma^{\rho(g_0)}(g_0)} \left|Q_s\widetilde{\widetilde{\varphi}}_2(h)\right|^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}\right)^{\frac{1}{2}},$$

where $\Gamma^{\rho(g_0)}(g_0) = \{(h,s)\in \mathbf{U}^n\colon\ |g_0^{-1}h| < \sqrt{s},\ s<\rho(g_0)^2\}$. Then A_3 is a finite constant and

$$\left|S_Q^L\varphi(g)-A_3\right| \leqslant \left(\int_{\rho(g_0)^2}^{\infty}\int_{|g^{-1}h|<\sqrt{s}} \left|Q_s^L\varphi(h)\right|^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}\right)^{\frac{1}{2}}$$

$$+\left|\left(\int_{\Gamma^{\rho(g_0)}(g)} \left|Q_s\varphi(h)\right|^2 \frac{dh\,ds}{s^{\frac{Q}{2}+1}}\right)^{\frac{1}{2}} - A_3\right|$$

$$
\begin{aligned}
&+\Bigg(\int\limits_{\Gamma^{\rho(g_0)}(g)} \big|Q_s^L\varphi(h)-Q_s\varphi(h)\big|^2\,\frac{dh\,ds}{s^{\frac{Q}{2}+1}}\Bigg)^{\frac{1}{2}}\\
&= S_1(g)+S_2(g)+S_3(g).
\end{aligned}
\tag{59}
$$

Similar to (54), we apply (52) to get

$$
\begin{aligned}
S_1(g)^2 &\leqslant \int\limits_{\rho(g_0)^2}^{\infty}\int\limits_{|g^{-1}h|<\sqrt{s}}\Bigg(\int\limits_{\mathbb{H}^n}\big|Q_s^L(h,w)\big|\big|\varphi(w)\big|\,dw\Bigg)^2\frac{dh\,ds}{s^{\frac{Q}{2}+1}}\\
&\leqslant C\int\limits_{\rho(g_0)^2}^{\infty}\Bigg(\int\limits_{\mathbb{H}^n}s^{-\frac{Q}{2}}e^{-As^{-1}|w^{-1}g|^2}\bigg(1+\frac{\sqrt{s}}{\rho(g)}\bigg)^{-2}\big|\varphi(w)\big|\,dw\Bigg)^2\frac{ds}{s}\\
&\leqslant C\|\varphi\|_{BMO_L}^2 \quad \text{for } g\in B(g_0,r).
\end{aligned}
\tag{60}
$$

To estimate $S_2(g)$, we apply triangle inequality to obtain

$$
\begin{aligned}
S_2(g) &\leqslant \Bigg(\int\limits_{\Gamma^{\rho(g_0)}(g)}\big|Q_s\widetilde{\widetilde{\varphi}}_1(h)\big|^2\frac{dh\,ds}{s^{\frac{Q}{2}+1}}\Bigg)^{\frac{1}{2}}+\Bigg(\int\limits_{\Gamma^{\rho(g_0)}(g)\Delta\Gamma^{\rho(g_0)}(g_0)}\big|Q_s\widetilde{\widetilde{\varphi}}_2(h)\big|^2\frac{dh\,ds}{s^{\frac{Q}{2}+1}}\Bigg)^{\frac{1}{2}}\\
&= S_{2,1}(g)+S_{2,2}(g),
\end{aligned}
\tag{61}
$$

where $E_1\Delta E_2=(E_1\setminus E_2)\cup(E_2\setminus E_1)$ denote the symmetric difference of two sets E_1 and E_2. Since S_Q is bounded on $L^2(\mathbb{H}^n)$ (cf. [9, Chapter 7]),

$$
\begin{aligned}
\frac{1}{|B|}\int\limits_B S_{2,1}(g)\,dg &\leqslant \frac{1}{|B|}\int\limits_B S_Q\widetilde{\widetilde{\varphi}}_1(g)\,dg\\
&\leqslant \Bigg(\frac{1}{|B|}\int\limits_B S_Q\widetilde{\widetilde{\varphi}}_1(g)^2\,dg\Bigg)^{\frac{1}{2}}\\
&\leqslant \Bigg(\frac{C}{|B^{**}|}\int\limits_{B^{**}}\big|\widetilde{\widetilde{\varphi}}_1(g)\big|^2\,dg\Bigg)^{\frac{1}{2}}\\
&\leqslant C\|\varphi\|_{BMO}.
\end{aligned}
\tag{62}
$$

We claim that

$$
S_{2,2}(g)\leqslant C\|\varphi\|_{BMO} \quad \text{for } g\in B(g_0,r). \tag{63}
$$

In fact,

$$
\begin{aligned}
S_{2,2}(g)^2 &\leqslant \int_0^{4r^2} \int_{B(g,\sqrt{s})\Delta B(g_0,\sqrt{s})} \biggl(\int_{(B^{**})^c} \bigl| Q_s\bigl(w^{-1}h\bigr)\bigr| \bigl|\widetilde{\widetilde{\varphi}}_2(w)\bigr|\, dw \biggr)^2 \frac{dh\, ds}{s^{\frac{Q}{2}+1}} \\
&\quad + \int_{4r^2}^{\rho(g_0)^2} \int_{B(g,\sqrt{s})\Delta B(g_0,\sqrt{s})} \biggl(\int_{(B^{**})^c} \bigl| Q_s(w^{-1}h)\bigr| \bigl|\widetilde{\widetilde{\varphi}}_2(w)\bigr|\, dw \biggr)^2 \frac{dh\, ds}{s^{\frac{Q}{2}+1}} \\
&= I_1 + I_2,
\end{aligned}
$$

where $I_2 = 0$ provided $2r \geqslant \rho(g_0)$. Note that, for $g \in B(g_0, r)$, $w \in (B^{**})^c$ and $|g^{-1}h| < \sqrt{s}$,

$$
\bigl| Q_s\bigl(w^{-1}h\bigr)\bigr| \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|w^{-1}g|^2} \leqslant Cs^{-\frac{Q}{2}} e^{-As^{-1}|w^{-1}g_0|^2}.
$$

Thus,

$$
\begin{aligned}
I_1 &\leqslant C \int_0^{4r^2} \int_{B(g,\sqrt{s})\Delta B(g_0,\sqrt{s})} \biggl(\int_{(B^{**})^c} s^{-\frac{Q}{2}} e^{-As^{-1}|w^{-1}g_0|^2} \bigl|\widetilde{\widetilde{\varphi}}_2(w)\bigr|\, dw \biggr)^2 \frac{dh\, ds}{s^{\frac{Q}{2}+1}} \\
&\leqslant C \int_0^{4r^2} \biggl(\int_{(B^{**})^c} \bigl|w^{-1}g_0\bigr|^{-(Q+1)} \bigl|\widetilde{\widetilde{\varphi}}_2(w)\bigr|\, dw \biggr)^2 ds \\
&\leqslant C\|\varphi\|_{BMO}^2 \quad \text{for } g \in B(g_0, r).
\end{aligned}
$$

Also note that

$$
\bigl|B(g,\sqrt{s})\Delta B(g_0,\sqrt{s})\bigr| \leqslant Crs^{\frac{Q-1}{2}} \quad \text{for } \sqrt{s} \geqslant 2r > \bigl|g^{-1}g_0\bigr|,
$$

which yields

$$
\begin{aligned}
I_2 &\leqslant \int_{4r^2}^{\rho(g_0)^2} \int_{B(g,\sqrt{s})\Delta B(g_0,\sqrt{s})} \biggl(\int_{(B^{**})^c} s^{-\frac{Q}{2}} e^{-As^{-1}|w^{-1}g_0|^2} \bigl|\widetilde{\widetilde{\varphi}}_2(w)\bigr|\, dw \biggr)^2 \frac{dh\, ds}{s^{\frac{Q}{2}+1}} \\
&\leqslant C \int_{4r^2}^{\rho(g_0)^2} r \biggl(\int_{(B^{**})^c} \bigl|w^{-1}g_0\bigr|^{-(Q+\frac{1}{4})} \bigl|\widetilde{\widetilde{\varphi}}_2(w)\bigr|\, dw \biggr)^2 \frac{ds}{s^{\frac{5}{4}}} \\
&\leqslant C\|\varphi\|_{BMO}^2 \quad \text{for } g \in B(g_0, r).
\end{aligned}
$$

This proves (63). Finally we estimate $S_3(g)$. By Lemma 4,

$$
\frac{\sqrt{s}}{\rho(h)} \leqslant C\biggl(1 + \frac{|g^{-1}h|}{\rho(g)}\biggr)^{m_0} \frac{\sqrt{s}}{\rho(g)}.
$$

For $g \in B(g_0, r)$ and $(h, s) \in \Gamma^{\rho(g_0)}(g)$, we have $|g^{-1}h| < \sqrt{s} < \rho(g_0) \sim \rho(g)$. It follows from (57) that

$$\left|G_s(h, w)\right| \leqslant C s^{-\frac{Q}{2}} e^{-As^{-1}|w^{-1}g|^2} \left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta}.$$

Similar to the estimation of $s_3(g)$, we get

$$\begin{aligned}
S_3(g)^2 &\leqslant \int_0^{\rho(g_0)^2} \int_{|g^{-1}h|<\sqrt{s}} \left(\int_{\mathbb{H}^n} \left|G_s(h, w)\right|\left|\varphi(w)\right| dw\right)^2 \frac{dh\, ds}{s^{\frac{Q}{2}+1}} \\
&\leqslant C \int_0^{\rho(g_0)^2} \left(\int_{\mathbb{H}^n} s^{-\frac{Q}{2}} e^{-As^{-1}|w^{-1}g|^2} \left(\frac{\sqrt{s}}{\rho(g)}\right)^{\delta} \left|\varphi(w)\right| dw\right)^2 \frac{ds}{s} \\
&\leqslant C\|\varphi\|_{BMO_L}^2 \quad \text{for } g \in B(g_0, r). \qquad (64)
\end{aligned}$$

By (59)–(64),

$$\frac{1}{|B|} \int_B \left|S_Q^L \varphi(g) - A_3\right| dg \leqslant C\|\varphi\|_{BMO_L}$$

which gives the estimate for S_Q^L. The proof of Theorem 6 is completed.

9. Results for stratified groups

In this section, we state results for stratified groups, which can be proved by the same argument for the Heisenberg group. We use the same notations and terminologies as in Folland and Stein's book [9].

Let G be a stratified group of dimension d with Lie algebra $\mathfrak{g}$. This means that $\mathfrak{g}$ is equipped with a family of dilations $\{\delta_r\colon r > 0\}$ and $\mathfrak{g}$ is a direct sum $\bigoplus_{j=1}^m \mathfrak{g}_j$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\mathfrak{g}_1$ generates $\mathfrak{g}$, and $\delta_r(X) = r^j X$ for $X \in \mathfrak{g}_j$. $Q = \sum_{j=1}^m j d_j$ is called the homogeneous dimension of G, where $d_j = \dim \mathfrak{g}_j$. G is topologically identified with $\mathfrak{g}$ via the exponential map $\exp\colon \mathfrak{g} \mapsto G$ and δ_r is also viewed as an automorphism of G. We fix a homogeneous norm of G, which satisfies the generalized triangle inequalities

$$\begin{aligned}
&|xy| \leqslant \gamma\big(|x| + |y|\big) \quad \text{for all } x, y \in G, \\
&\big||xy| - |x|\big| \leqslant \gamma|y| \quad \text{for all } x, y \in G \text{ with } |y| \leqslant \frac{|x|}{2},
\end{aligned}$$

where $\gamma \geqslant 1$ is a constant. The ball of radius r centered at x is written by

$$B(x, r) = \left\{y \in G\colon \left|x^{-1}y\right| < r\right\}.$$

The Haar measure on G is simply the Lebesgue measure on $\mathbb{R}^d$ under the identification of G with $\mathfrak{g}$ and the identification of $\mathfrak{g}$ with $\mathbb{R}^d$, where $d = \sum_{j=1}^m d_j$. The measure of $B(x,r)$ is

$$\left|B(x,r)\right| = br^Q,$$

where b is a constant.

We identify $\mathfrak{g}$ with $\mathfrak{g}_L$, the Lie algebra of left-invariant vector fields on G. Let $\{X_j\colon j = 1,\dots,d_1\}$ be a basis of $\mathfrak{g}_1$. The sub-Laplacian Δ_G is defined by

$$\Delta_G = \sum_{j=1}^{d_1} X_j^2.$$

Consider the Schrödinger operator $L = -\Delta_G + V$, where the potential V is nonnegative and belongs to the reverse Hölder class $B_{Q/2}$. The Hardy space $H_L^1(G)$ associated with the Schrödinger operator L is defined by the maximal function with respect to the semigroup $\{T_s^L\colon s>0\} = \{e^{-sL}\colon s>0\}$ (cf. [14]). A function $f \in L^1(G)$ is said to be in $H_L^1(G)$ if the maximal function $T_L^* f$ belongs to $L^1(G)$, where $T_L^* f(x) = \sup_{s>0} |T_s^L f(x)|$. The norm of such a function is defined by $\|f\|_{H_L^1} = \|T_L^* f\|_{L^1}$.

Let φ be a locally integrable function on G. $B = B(x,r)$ is a ball centered at x. Set

$$\varphi(B) = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \varphi(y)\,dy$$

and

$$\varphi(B,V) = \begin{cases} \varphi(B), & \text{if } r < \rho(x), \\ 0, & \text{if } r \geqslant \rho(x), \end{cases}$$

where the auxiliary function $\rho(x) = \rho(x,V)$ is defined as before; that is,

$$\rho(x) = \sup_{r>0} \left\{ r\colon \frac{1}{r^{Q-2}} \int\limits_{B(x,r)} V(y)\,dy \leqslant 1 \right\}, \quad x \in G.$$

Define the revised sharp function related to the potential V by

$$\varphi_V^\sharp(x) = \sup_{x\in B} \frac{1}{|B|} \int\limits_B \left|\varphi(y) - \varphi(B,V)\right| dy.$$

The space $BMO_L(G)$ associated with the Schrödinger operator L is defined as follows.

Definition 1′. Let φ be a locally integrable function on G. If $\varphi_V^\sharp \in L^\infty(G)$, then we say $\varphi \in BMO_L(G)$ and set $\|\varphi\|_{BMO_L} = \|\varphi_V^\sharp\|_{L^\infty}$.

If $\varphi \in BMO_L(G)$, then

$$\int_G \frac{|\varphi(x)|}{(1+|x|)^{Q+1}}\,dx < \infty. \tag{65}$$

Let L_c^∞ denote the space of all bounded functions with compact supports and set

$$\mathcal{L}_\varphi(f) = \int_G f(x)\varphi(x)\,dx, \quad f \in L_c^\infty,\ \varphi \in L^1_{\mathrm{loc}}(G). \tag{66}$$

Theorem 1′.

(a) *Suppose $\varphi \in BMO_L(G)$. Then $\mathcal{L}_\varphi$ given by* (66) *extends to a bounded linear functional on $H_L^1(G)$ and satisfies*

$$\|\mathcal{L}_\varphi\| \leqslant C\|\varphi\|_{BMO_L}.$$

(b) *Conversely, every bounded linear functional $\mathcal{L}$ on $H_L^1(G)$ can be realized as $\mathcal{L} = \mathcal{L}_\varphi$ with $\varphi \in BMO_L(G)$ and*

$$\|\varphi\|_{BMO_L} \leqslant C\|\mathcal{L}\|.$$

Let us consider the *adjoint Riesz transforms* $\widetilde{R}_j^L$ for the Schrödinger operator L defined by

$$\widetilde{R}_j^L = L^{-\frac{1}{2}}X_j, \quad j = 1, \dots, d_1.$$

Theorem 2′.

(a) *$\widetilde{R}_j^L$ are bounded on $BMO_L(G)$.*
(b) *$\varphi \in BMO_L(G)$ if and only if there exist $\varphi_0, \varphi_1, \dots, \varphi_{d_1} \in L^\infty(G)$ such that*

$$\varphi(x) = \varphi_0(x) + \sum_{j=1}^{d_1} \widetilde{R}_j^L \varphi_j(x).$$

Let S be the semidirect extension of G by the one parameter group of dilations. The group law of S is given by

$$(x,r)(y,s) = \bigl(x(\delta_r y), rs\bigr), \quad x, y \in G,\ r, s > 0.$$

If G is an H-type group, then S is known as an *NA* group or a Damek–Ricci space (cf. [6]), which has been investigated extensively. For any ball $B = B(x,r)$ in G, we define the *Carleson box* $\Omega(B) = \Omega(x,r)$ based on B by

$$\Omega(x,r) = \bigl\{(y,s) \in S\colon\ \bigl|x^{-1}y\bigr| < r,\ 0 < s < r\bigr\}.$$

A nonnegative Borel measure μ on $\mathbf{U}^n$ is called a *Carleson measure* if

$$\|\mu\|_{\mathcal{C}} = \sup_B \frac{\mu(\Omega(B))}{|B|} < \infty.$$

Let

$$Q_r^L \varphi(x) = r^2 \left(\frac{d}{ds} T_s^L \bigg|_{s=r^2} \varphi \right)(x), \quad (x,r) \in S.$$

Then $Q_r^L \varphi$ is well defined if φ satisfies (65) and we obtain a nonnegative Borel measure $d\mu_\varphi$ on S defined by

$$d\mu_\varphi(x,r) = \left| Q_r^L \varphi(x) \right|^2 \frac{dg\, dr}{r}, \quad (x,r) \in S.$$

Theorem 3′.

(a) *If $\varphi \in BMO_L(G)$, then $d\mu_\varphi$ is a Carleson measure with*

$$\|d\mu_\varphi\|_{\mathcal{C}} \leqslant C \|\varphi\|^2_{BMO_L}.$$

(b) *Conversely, if φ satisfies* (65) *and $d\mu_\varphi$ is a Carleson measure, then $\varphi \in BMO_L(G)$ and*

$$\|\varphi\|^2_{BMO_L} \leqslant C \|d\mu_\varphi\|_{\mathcal{C}}.$$

We also define the Littlewood–Paley function s_Q^L and the Lusin area integral S_Q^L related to L respectively by

$$s_Q^L f(x) = \left(\int_0^\infty \left| Q_r^L f(x) \right|^2 \frac{dr}{r} \right)^{\frac{1}{2}}$$

and

$$S_Q^L f(x) = \left(\int_0^\infty \int_{|x^{-1}y|<r} \left| Q_r^L f(y) \right|^2 \frac{dy\, dr}{r^{Q+1}} \right)^{\frac{1}{2}}.$$

Theorem 4′. *The operators s_Q^L and S_Q^L are bounded from $H_L^1(G)$ to $L^1(G)$ and bounded from $L^1(G)$ to $L^{1,\infty}(G)$. For $1 < p < \infty$,*

$$\left\| s_Q^L f \right\|_{L^p} \sim \left\| S_Q^L f \right\|_{L^p} \sim \|f\|_{L^p}.$$

The duality inequality of tent spaces can also be extended to S. Let $F(x,r)$ and $\Phi(x,r)$ be measurable functions on S. We set

$$\mathcal{A}(F)(x) := \left(\int_0^\infty \int_{|x^{-1}y|<r} \left|F(y,r)\right|^2 \frac{dy\,dr}{r^{Q+1}} \right)^{\frac{1}{2}},$$

$$\mathcal{C}(\Phi)(g) := \sup_{x\in B} \left(\frac{1}{|B|} \int_{\Omega(B)} \left|\Phi(y,r)\right|^2 \frac{dy\,dr}{r} \right)^{\frac{1}{2}}.$$

Theorem 5′. *Let $F(x,r)$ and $\Phi(x,r)$ be measurable functions on S such that $\mathcal{A}(F) \in L^1(S)$ and $\mathcal{C}(\Phi) \in L^\infty(S)$. Then we have the following duality inequality.*

$$\begin{aligned} \int_S \left|F(x,r)\Phi(x,r)\right| \frac{dx\,dr}{r} &\leqslant C \int_G \mathcal{A}(F)(x)\mathcal{C}(\Phi)(x)\,dx \\ &\leqslant C \left\|\mathcal{A}(F)\right\|_{L^1} \left\|\mathcal{C}(\Phi)\right\|_{L^\infty}. \end{aligned}$$

We also have

Theorem 6′. *The Hardy–Littlewood maximal function M and the semigroup maximal function T_L^* are bounded on $BMO_L(G)$. The Littlewood–Paley function s_Q^L and the Lusin area integral S_Q^L are bounded on $BMO_L(G)$.*

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